Compressible flow (Burgers' equation)

Problem formulation

1D Cauchy problem

$$\begin{cases} u_t + f_x(u) = 0 \\ u(x,0) = u_0(x) \end{cases}$$

where $f(u)$ is a non-linear function of the unknown. Inserting the Burgers' equation $f(u) = u^2/2$ leads to

$$\begin{cases} u_t + (\frac{u^2}{2})_x = 0 \\ u(x,0) = u_0(x) \end{cases}$$

or written in convective form:

$$\begin{cases} u_t + uu_x = 0 \\ u(x,0) = u_0(x) \end{cases}$$

Task

Complete the code to solve the problem with the Newton-Raphson scheme. Explain the main changes in the code. Compare and discuss the results.

The provided MATLAB code came with an implementation of an explicit forward Euler scheme as well as an implicit backward Euler scheme which applies the Picard method to solve the non-linear system of equations. The given implementation of the Picard method is used as a skeleton for introducing the Newton-Raphson algorithm. Therefore, the following paragraph will only highlight the changes and differences w.r.t. to the Picard method.

The initial condition for both methods reads as follows:

$$^0\mathbf{U}^{\mathbf{n+1}} = \mathbf{U}^n.$$

The Picard method solves the following system of equations iteratively

$$^{k+1}\mathbf{U}^{\mathbf{n+1}} = \mathbf{A}^{-1}(^k\mathbf{U}^{\mathbf{n+1}})(\mathbf{MU}),$$

where $\mathbf{A}(\mathbf{U}^{\mathbf{n+1}})$ is

$$\mathbf{A}(\mathbf{U}^{\mathbf{n+1}}) = (\mathbf{M} + \Delta t(\mathbf{C}(\mathbf{U}^{n+1}) + \epsilon\mathbf{K})).$$

The main difference between the Newton-Raphson and the Picard scheme is the computation of the Jacobian of the residual. The residual has to be solved at every time step and takes the form

$$\mathbf{f}(\mathbf{U}) = \mathbf{AU} - (\mathbf{MU})$$

with $\mathbf{A}$ as described above.

In every time step k is iterated until convergence

$$^{k+1}\mathbf{U}^{\mathbf{n+1}} =^{k} \mathbf{U}^{\mathbf{n+1}} - \mathbf{J}^{-1}(^{k}\mathbf{U}^{\mathbf{n+1}})\mathbf{f}(^{k}\mathbf{U}^{\mathbf{n+1}})$$

where the Jacobian is

$$\mathbf{J} = \frac{d\mathbf{f}}{d\mathbf{U}} = \mathbf{A}(\mathbf{U}) + \frac{\delta \mathbf{C}(\mathbf{U})}{\delta \mathbf{U}}\mathbf{U}.$$

As $\mathbf{A}$ is already implemented for the Picard method the residual $\mathbf{f}(\mathbf{U})$ can be easily computed. The difficult part is the computation of the derivative of the convective matrix w.r.t. $\mathbf{U}$.

Eventually, the derivative results in

$$\frac{\delta \mathbf{C}(\mathbf{U})}{\delta \mathbf{U}}\mathbf{U} = \Delta t \mathbf{C}(\mathbf{U})$$

and can be added to the matrix $\mathbf{A}$ to complete the Jacobian. We obtain

$$\mathbf{J} = \mathbf{M} + \Delta t(2\mathbf{C}(\mathbf{U}^{n+1}) + \epsilon \mathbf{K}).$$

The main changes that have to be introduced to convert the Picard algorithm into the Newton-Raphson scheme can be divided into the two steps mentioned above. Firstly, compute the residual with the help of matrix $\mathbf{A}$. Secondly, find the derivative of the convective matrix w.r.t. $\mathbf{U}$ and use it to compute the Jacobian.

In order to compare the stability of the results of the different methods four cases with a decreasing initial condition will be investigated:

- **case 1:** $\Delta t = 0.005, \epsilon = 1e^{-2}$
- **case 2:** $\Delta t = 0.05, \epsilon = 1e^{-2}$
- **case 3:** $\Delta t = 0.1, \epsilon = 1e^{-2}$
- **case 4:** $\Delta t = 0.005, \epsilon = 1e^{-4}$

The initial condition is shown in Figure 1. The time frame for all cases is $t \in [0, 4]$.

The solutions obtained for case 1 are presented on the left-hand side of Figure 2 and accordingly the results for case 2 can be found on the right-hand side. In the first case the explicit scheme as well as both implicit schemes (Picard & Newton-Raphson) produce smooth and accurate results over the whole domain and differences are hardly visible. This changes for the second case where the size of the time step was increased by a factor of 10 compared to the first case. Although both implicit schemes still show a smooth solution, the explicit scheme starts oscillating heavily after a few time steps.

When increasing the size of the time step even more a picture similar to the second case arises. The implicit methods can approximate the solution accurately but the explicit scheme shows

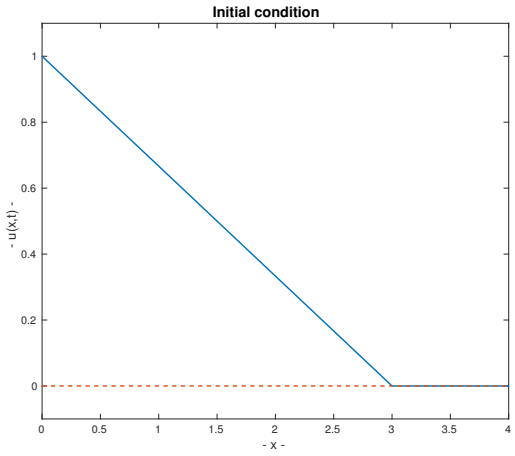

Figure 1: Decreasing initial condition

spurious oscillations. This can be observed in the third case displayed on the left-hand side of Figure 3. Instead of increasing the step size, in the fourth case ϵ is reduced by the order of two while the step size is kept at $\Delta t = 0.005$. Once again the explicit method fails to approximate the solution showing oscillation after several time steps. It is interesting to note that in the fourth case also the implicit Picard and Newton-Raphson scheme start to oscillate around the discontinuity. As known from the lecture when $\lim_{\epsilon \to 0}$ the solution of the original hyperbolic problem is obtained. To gain a valid solution for the chosen ϵ the step size and iteration accuracy must be adjusted accordingly which would result in unreasonable computational effort. https://www.overleaf.com/project/5c8bf1e39ebbc67762d1106d

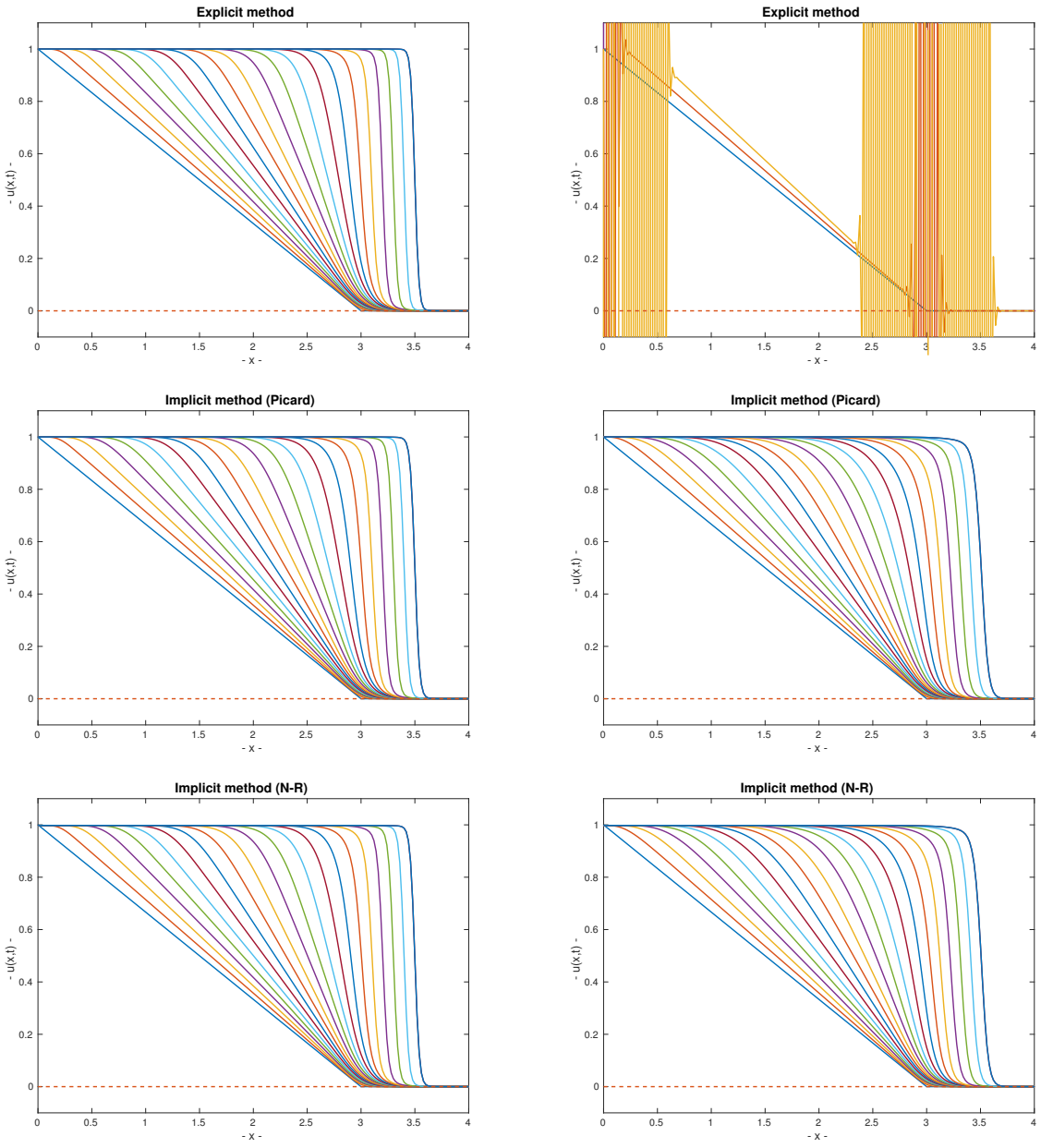

Figure 2: Case 1 (left) with $\Delta t = 0.005, \epsilon = 1e^{-2}$ and case 2 (right) with $\Delta t = 0.05, \epsilon = 1e^{-2}$

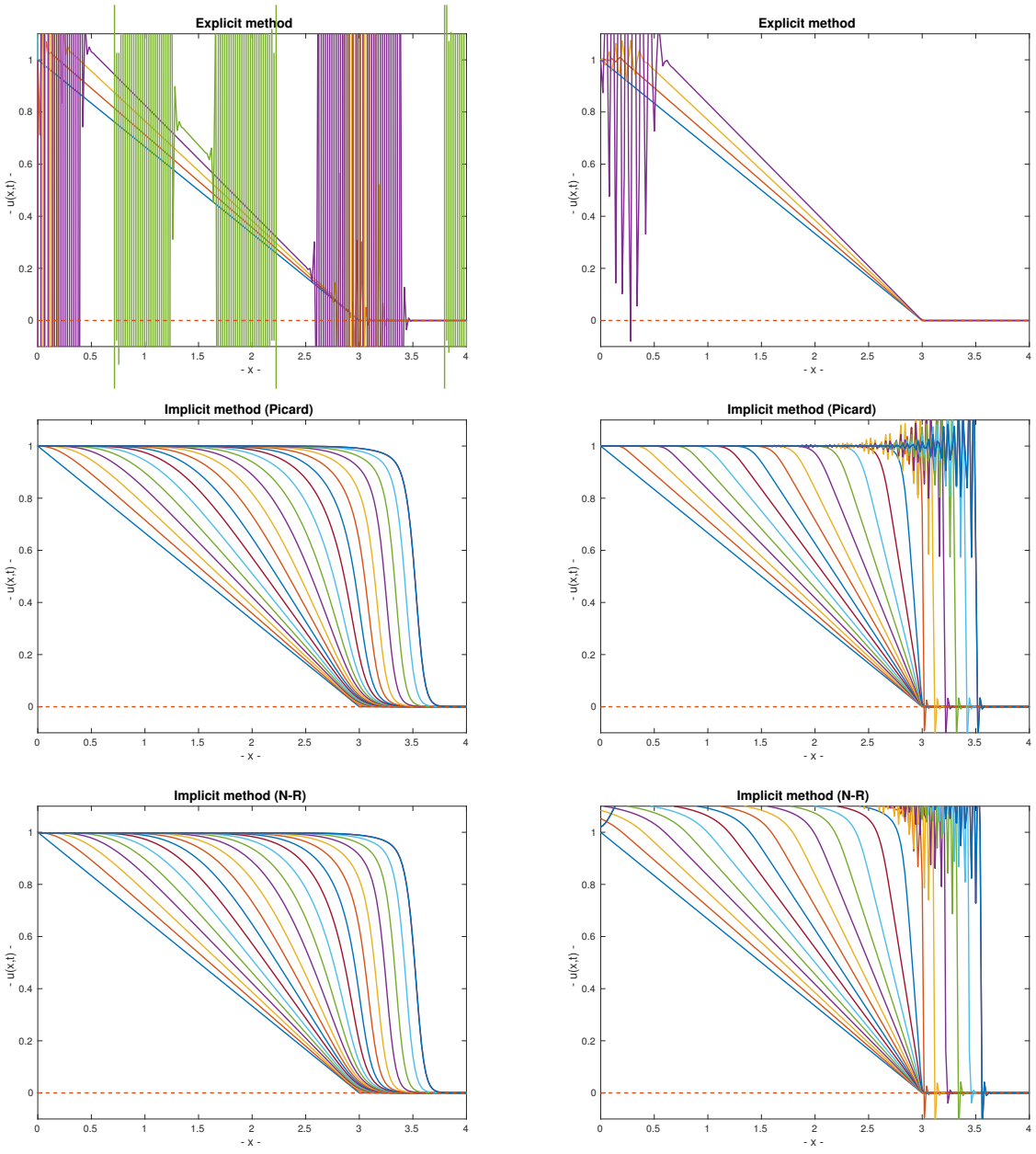

Figure 3: Case 1 (left) with $\Delta t = 0.1, \epsilon = 1e^{-2}$ and case 2 (right) with $\Delta t = 0.005, \epsilon = 1e^{-4}$