Navier-Stokes Equation

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Introduction

Navier-Stokes equation rules the motion of viscous, incompressible, Newtonian fluids which represents the bast majority of fluid problems in our every day live, such as ocean current, atmospheric dynamics, air flow around a wing, a car etc. A characteristic parameter is the so called Reynolds number (Re), which accounts for the relation between inertia and viscous forces. For highly viscous fluids, Re is really low and inertia terms can be neglected, simplifying the equation to be solved in the so called Stokes Equation.

In this work, Stokes equation will be discretized and solved computationally, studying its behaviour for different elements. Later, inertia terms will be introduced and Navier-Stokes equation will be recovered, discretized and solved with MatLab.

Problem statement (Stokes)

Stokes equation weak form is obtained via Weighted Residuals and discretized by using Galerkin methods as follows:

Stokes discretization

$$\begin{cases} -\nu \nabla^2 \vec{v} + \nabla p = \vec{b} & \text{in } \Omega \\ \nabla \cdot \vec{v} = 0 & \text{in } \Omega \\ v = v_0 & \text{in } \partial\Omega \end{cases}$$

Stokes equation + incompressibility

weighted residual method:

$$\begin{cases} -\int_\Omega \vec{w}\, \nu \nabla^2 \vec{v}\, d\Omega + \int_\Omega \nabla p \cdot \vec{w}\, d\Omega = \int_\Omega \vec{w} \cdot \vec{b} \\ \int_\Omega q\, \nabla \cdot \vec{v}\, d\Omega = 0 \end{cases}$$

Knowing that

$$\nabla (\vec{w}\, \nu \nabla \vec{v}) = \nabla \vec{w} : \nu \nabla \vec{v} + \vec{w}\, \nu \nabla^2 \vec{v} \quad \text{then}$$

$$\int \vec{w}\, \nu \nabla^2 \vec{v}\, d\Omega = \int \nabla(\vec{w} \cdot \nu \nabla \vec{v})\, d\Omega - \int \nabla \vec{w} : \nu \nabla \vec{v}\, d\Omega$$

divergence theorem

$$\int_\Omega \nabla(\vec{w} \cdot \nu \nabla \vec{v})\, d\Omega = \int_\Gamma (\vec{w} \cdot \nu \nabla \vec{v}) \hat{n}\, d\Gamma \quad \text{given that } \vec{w} = 0 \text{ on } \partial\Omega$$

then
first equation reads:

$$\int \nabla\vec{w} : \nu \nabla\vec{v}\, d\Omega + \int_\Omega \nabla p \cdot \vec{w}\, d\Omega = \int_\Omega \vec{w} \cdot \vec{b}\, d\Omega$$

Again integrating by parts pressure terms read:

$$\int_\Omega \nabla p \cdot \vec{w}\, d\Omega = -\int p\, \nabla\cdot\vec{w}\, d\Omega + \cancel{\int_\Gamma \vec{w}\, \nabla p \cdot d\Gamma}$$

Then Weighted Residual methods reads:

$$\begin{cases} \int_\Omega \nabla\vec{w} : \nu \nabla\vec{v}\, d\Omega - \int_\Omega p\, \nabla\cdot\vec{w}\, d\Omega = \int \vec{w} \cdot \vec{b}\, d\Omega & \forall\, \vec{w} \in V \\ \int_\Omega q\, \nabla\cdot\vec{v}\, d\Omega = 0 \end{cases}$$

Now we make an approximation of our unknown functions $\vec{v}$ and p

$$\vec{v}(\vec{x}) \simeq \vec{v}^h(\vec{x}) = \sum_{i=1}^{m} N_i(\vec{x})\, \vec{v}_i = \sum_{i=1}^{m} N_i(\vec{x}) \begin{bmatrix} v_x^i \\ v_y^i \end{bmatrix}$$

$$p(\vec{x}) \simeq p^h(\vec{x}) = \sum_{j=1}^{m} \hat{N}_j(\vec{x})\, p_j$$

using as a test functions $(\vec{w}, q)$ the shape functions then

$$\begin{cases} \int \nabla N_i : \nu \nabla N_j\, d\Omega\, \vec{v}_j - \int p\, \nabla\cdot N_i\, d\Omega = \int N_i\, \vec{b}\, d\Omega \\ \int q\, \nabla\cdot\vec{v}\, d\Omega = 0 \end{cases}$$

In matrix form it can be written as:

$$\begin{pmatrix} \overline{\overline{K}} & \overline{\overline{G}}^T \\ \overline{\overline{G}} & \overline{\overline{0}} \end{pmatrix} = \begin{pmatrix} \vec{f} \\ 0 \end{pmatrix}$$

That enables us to solve the Cavity problem which consist in a square domain of dimension one by one with impermeable walls except the one on the top.

Results

Simulations done with MatLab provided the solution of the velocity and pressure, which allows to plot the velocity field and pressure field. Stream lines have been also plotted, all for different element types.

Using quadrilateral elements, quadratic for velocity and linear for pressure (Q2Q1), the mesh for 10x10 elements in *Figure 1.* is obtained.

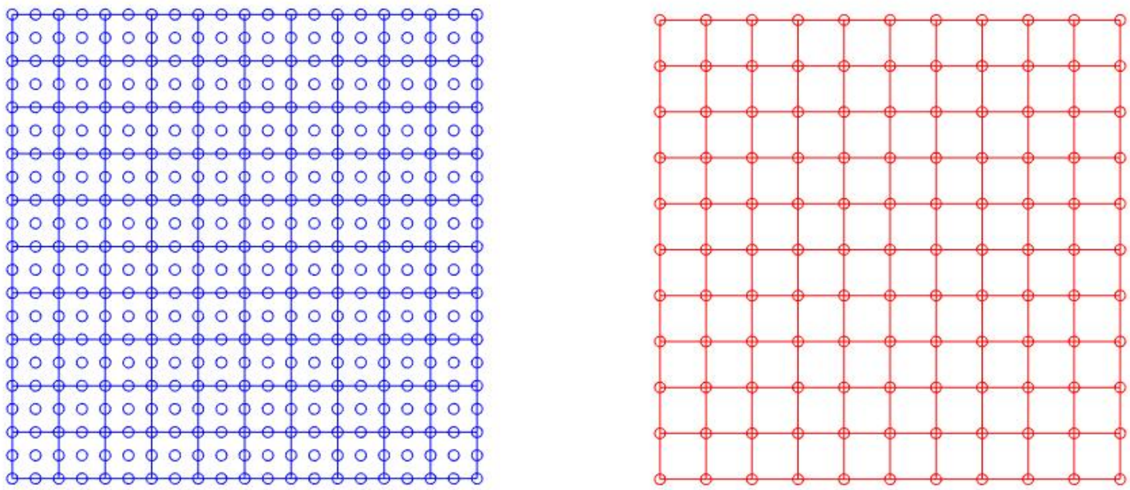

Figure 1. Mesh of the domain for the velocity (left) and for the pressure (right).

With that mesh, the solution is:

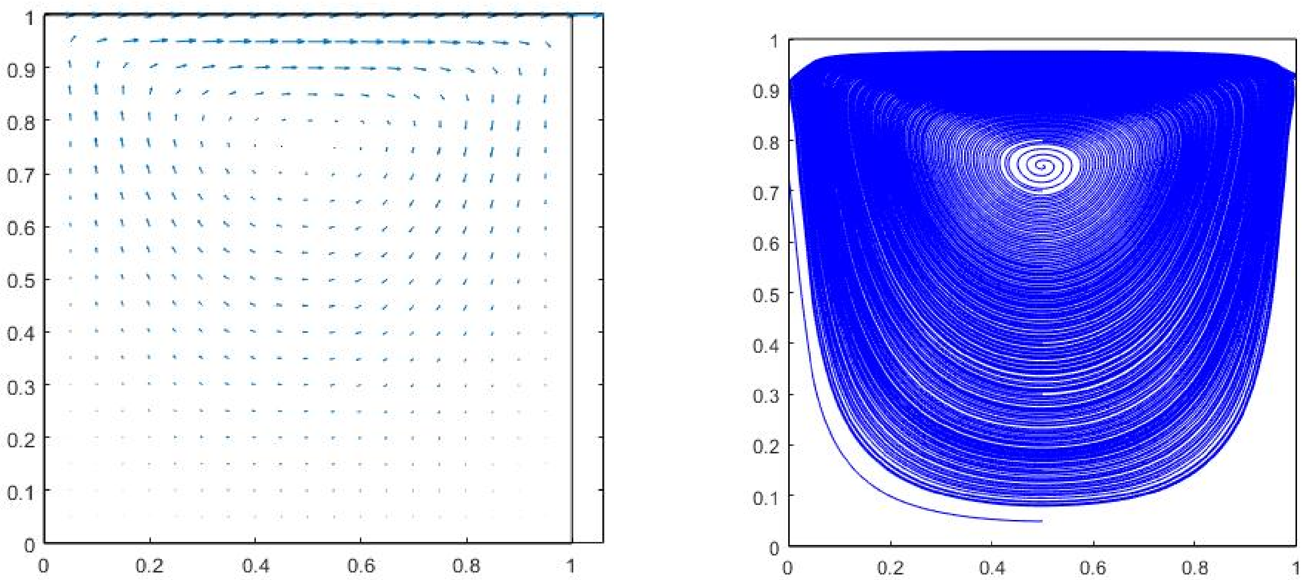

Figure 2. Velocity field represented by vectors (left) and streamlines (right) for Q2Q1 elements.

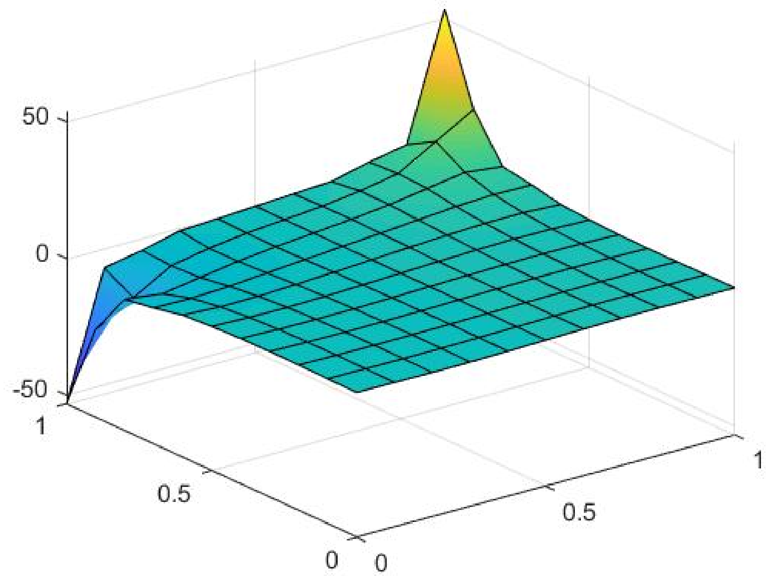

Figure 3. Pressure distribution over the domain.

If instead of Q2Q1 elements, Q1Q1 elements are used, that is linear for velocity and pressure. Then:

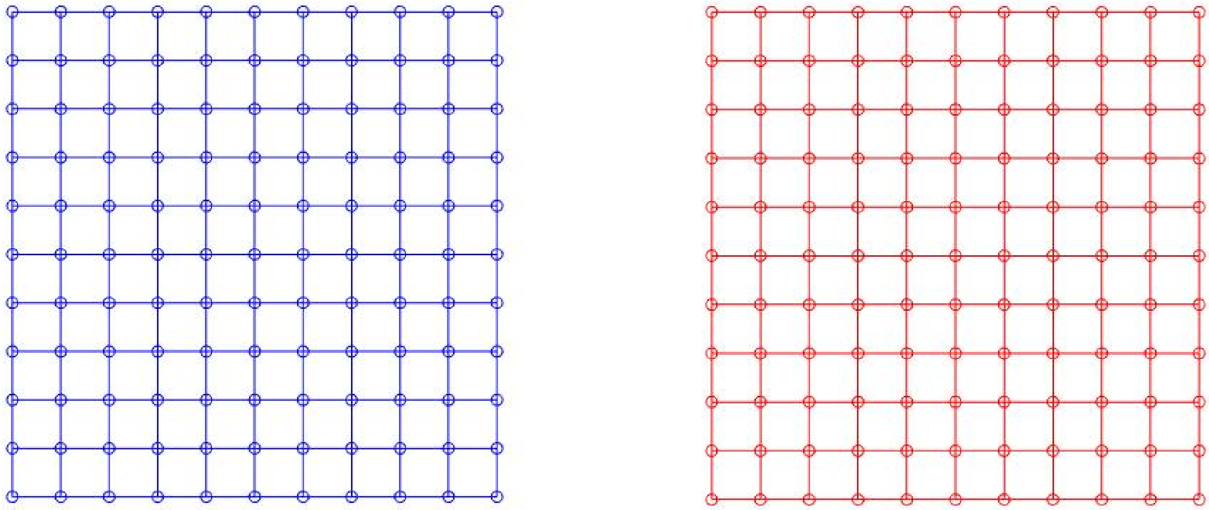

Figure 4. Mesh for the velocity (left) and for the pressure (right) with linear quadrilateral elements.

With Q1Q1 elements the solution obtained is the following:

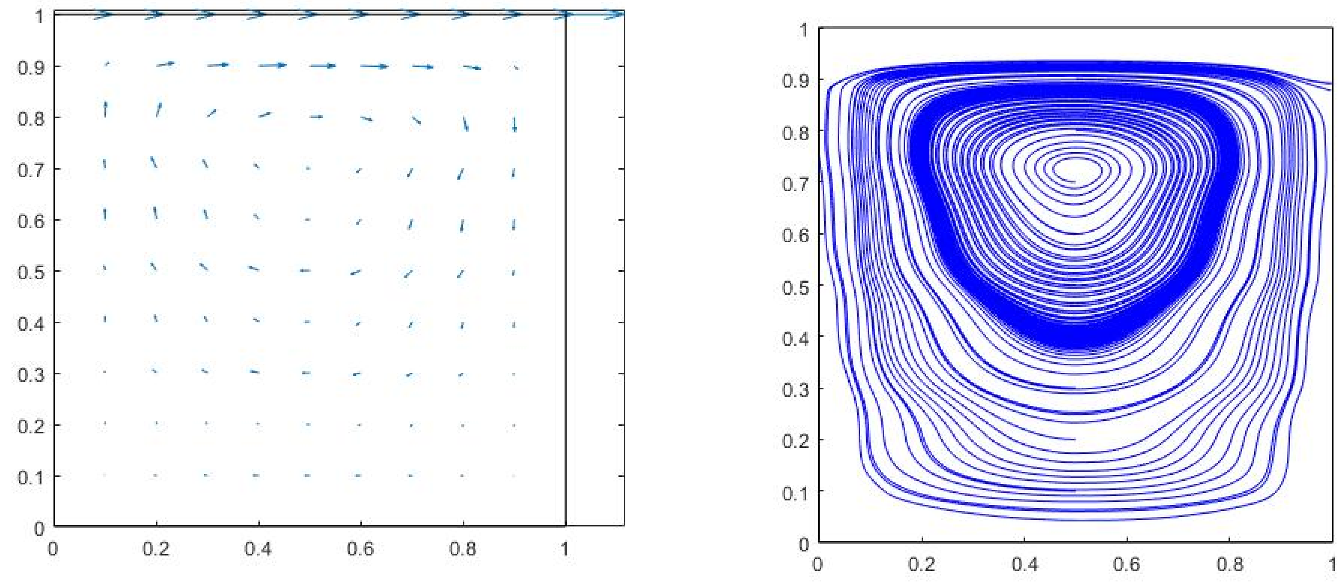

Figure 5. Velocity field represented by vectors (left) and streamlines (right) for Q1Q1 elements.

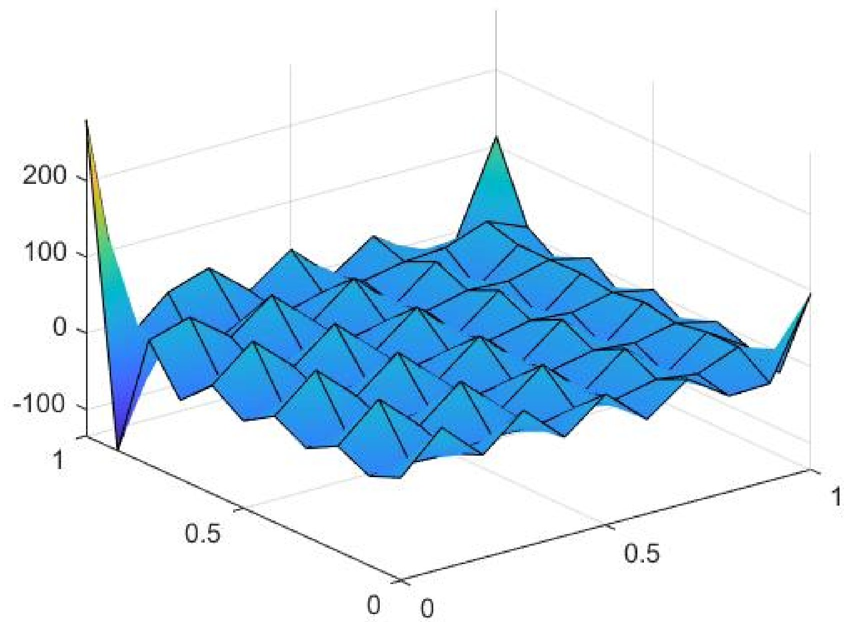

Figure 6. Pressure distribution over the domain.

Comparing Q2Q1 elements with Q1Q1, it is clearly seen what Q2Q1 provide good solution. However, Q1Q1 elements does not perform well due to the existence of oscillation in pressure. Since pressure and velocity are coupled, those oscillations also affect the velocity field.
This behaviour is also seen when using triangular elements. In the following figures, quadratic triangular elements for velocity and triangular linear elements for velocity are used:

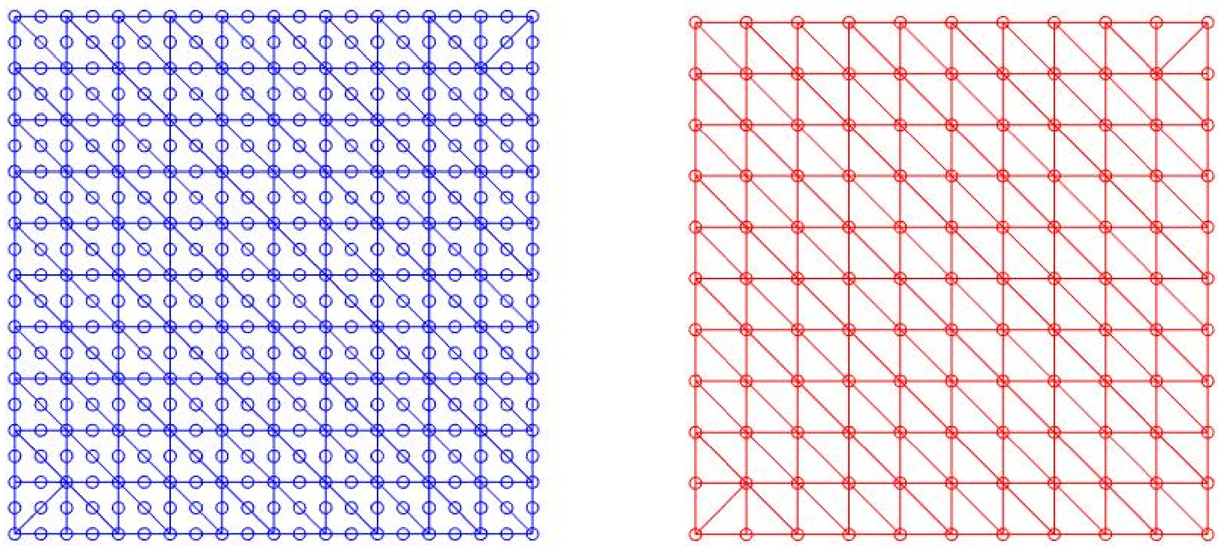

Figure 7. Mesh for the velocity with quadratic triangular elements (left) and for the pressure (right) with linear triangular elements.

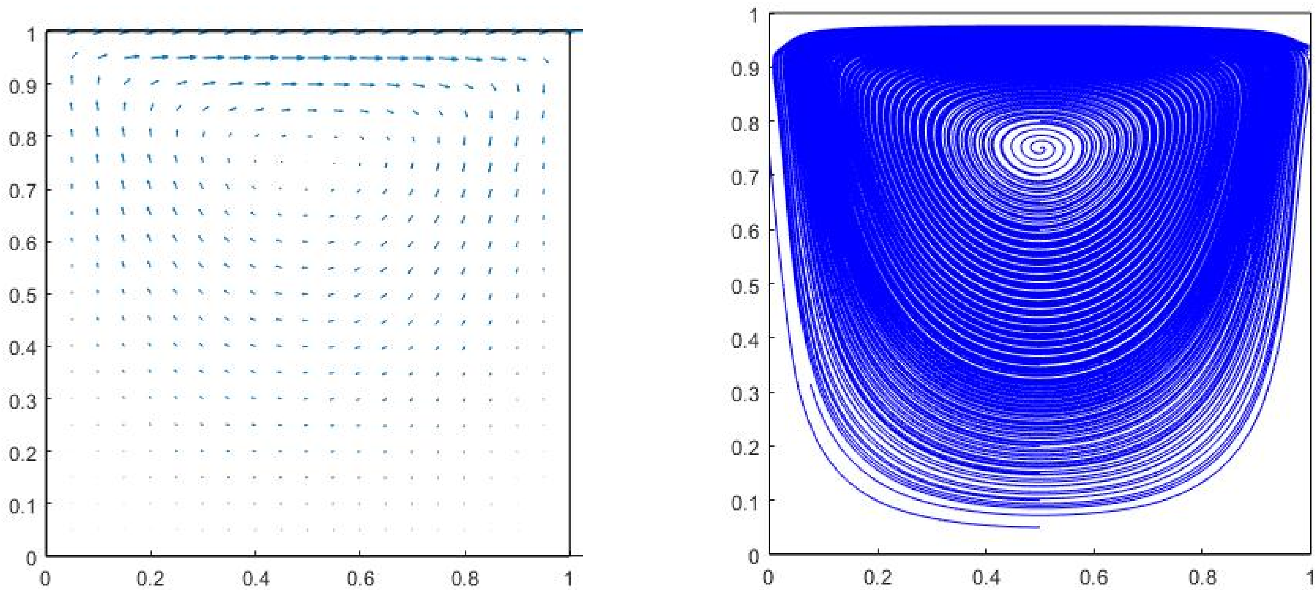

Figure 8. Velocity field represented by vectors (left) and streamlines (right).

Comparing *Figure 8.*with *Figure 2.* is observed that the solution obtained is good enough and represents a good solution to the cavity problem. Regarding pressure, from *Figure 9.* it is also seen that the pressure doesn't present oscillations.

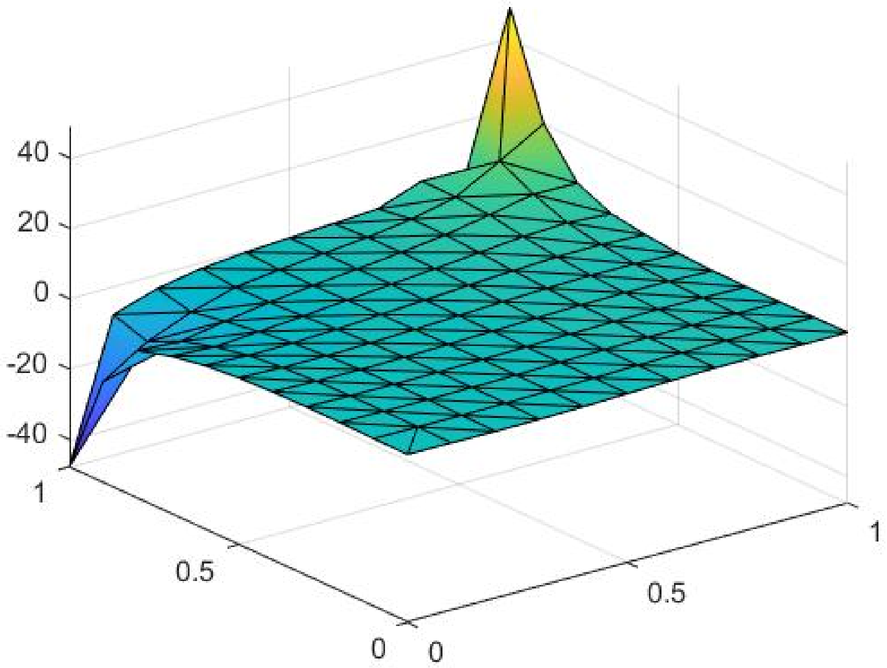

Figure 9. Pressure distribution over the domain.

Doing the same but with linear triangles for both velocity and pressure:

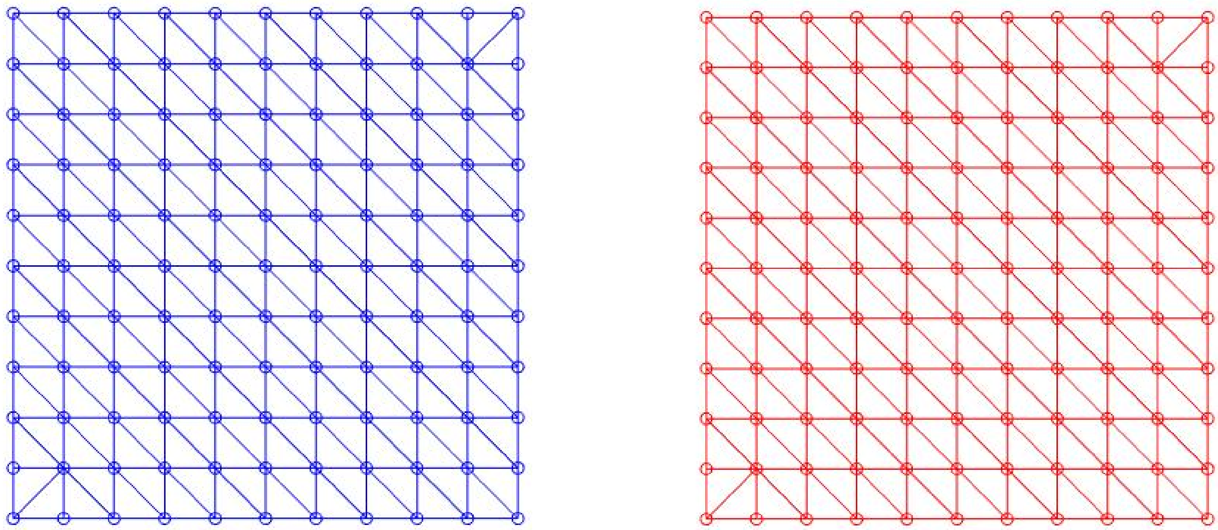

Figure 10. Mesh for the velocity with linear triangular elements (left) and for the pressure (right) with linear triangular elements (P1P1).

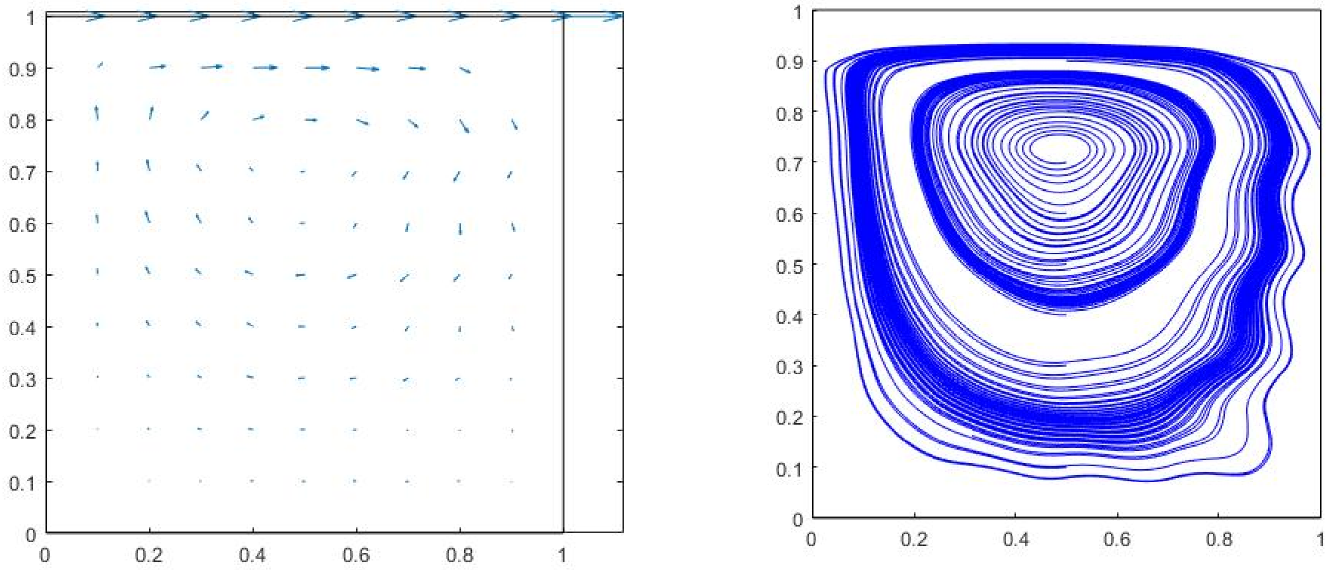

Figure 11. Velocity field represented by vectors (left) and streamlines (right).

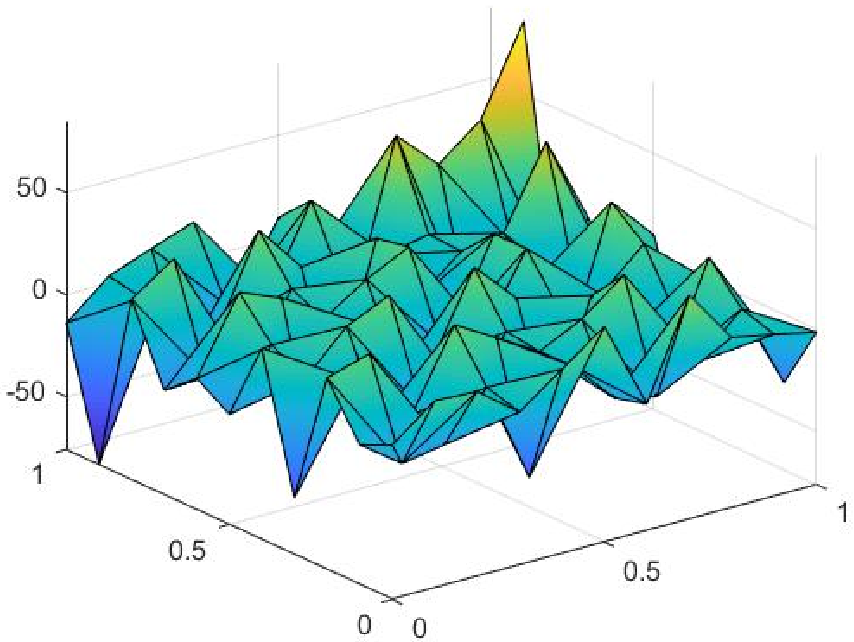

Figure 12. Pressure distribution over the domain.

Again, oscillations appear and the solution doesn't represent the real behaviour of the system.
The reason for which oscillations appear for some elements is that those elements doesn't fulfil the LBB (Ladyzhenskaya-Babuška-Brezzi) condition. For elements which are not stable because don't fulfil LBB condition, stabilization techniques must be used. In our case, Galerkin Least Square stabilization has been used.

This method consist in adding a new term:

$$\sum_e \int_{\Omega_e} \boldsymbol{\tau}\boldsymbol{\mathcal{L}}(\boldsymbol{w}, q) \cdot \Big(\boldsymbol{\mathcal{L}}(\boldsymbol{v}, p) - \boldsymbol{\mathcal{F}}\Big) d\Omega$$

Which will result in:

$$\begin{pmatrix} \mathbf{K} + \overline{\mathbf{K}} & \mathbf{G}^T + \overline{\mathbf{G}}^T \\ -\mathbf{G} + \overline{\mathbf{G}} & \mathbf{0} + \overline{\mathbf{L}} \end{pmatrix} \begin{pmatrix} \mathbf{v} \\ \mathbf{p} \end{pmatrix} = \begin{pmatrix} \mathbf{f} + \overline{\mathbf{f}}_w \\ \mathbf{0} + \overline{\mathbf{f}}_q \end{pmatrix}$$

$$\begin{aligned}
\overline{\mathbf{K}} &\leftarrow \sum_e \int_{\Omega_e} \tau_1(-\nu\nabla^2\boldsymbol{w}) \cdot (-\nu\nabla^2\boldsymbol{v}) d\Omega + \sum_e \int_{\Omega_e} \tau_2(\boldsymbol{\nabla}\cdot\boldsymbol{w})(\boldsymbol{\nabla}\cdot\boldsymbol{v}) d\Omega \\
\overline{\mathbf{G}} &\leftarrow \sum_e \int_{\Omega_e} \tau_1(\boldsymbol{\nabla} q) \cdot (-\nu\nabla^2\boldsymbol{v}) d\Omega \\
\overline{\mathbf{L}} &\leftarrow \sum_e \int_{\Omega_e} \tau_1(\boldsymbol{\nabla} q) \cdot (\boldsymbol{\nabla} p) d\Omega \\
\overline{\mathbf{f}}_w &\leftarrow \sum_e \int_{\Omega_e} \tau_1(-\nu\nabla^2\boldsymbol{w}) \cdot (-\boldsymbol{f}) d\Omega \\
\overline{\mathbf{f}}_q &\leftarrow \sum_e \int_{\Omega_e} \tau_1(\boldsymbol{\nabla} q) \cdot (-\boldsymbol{f}) d\Omega
\end{aligned}$$

Although GLS method results in all those new methods, only L and f_q have been implemented. The reason is that we are only interested to stabilize linear elements since quadratic elements are stable. When considering linear elements, G,f_w vanishes due to the second derivative, the first term of K also vanishes for the same reason and the second term is zero due to τ_2=0 for linear elements in order to guarantee stability and convergence.

That being said, the results for Q1Q1 elements are:

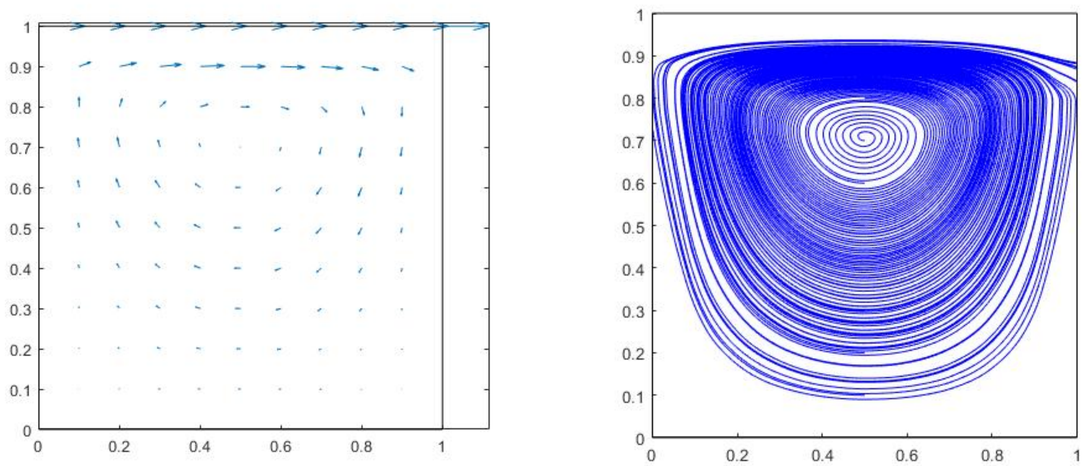

Figure 13. Velocity field represented by vectors (left) and streamlines (right) for Q1Q1 elements.

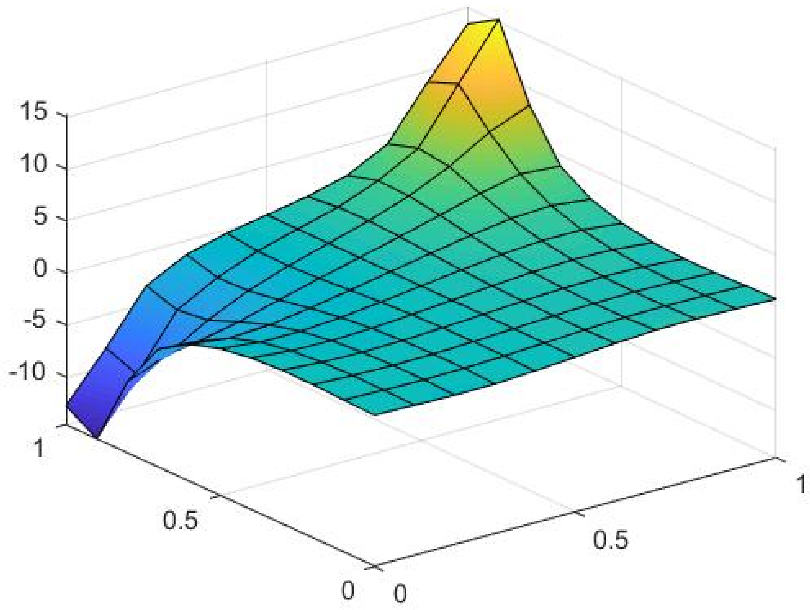

Figure 14. Pressure distribution over the domain.

For P1P1 elements:

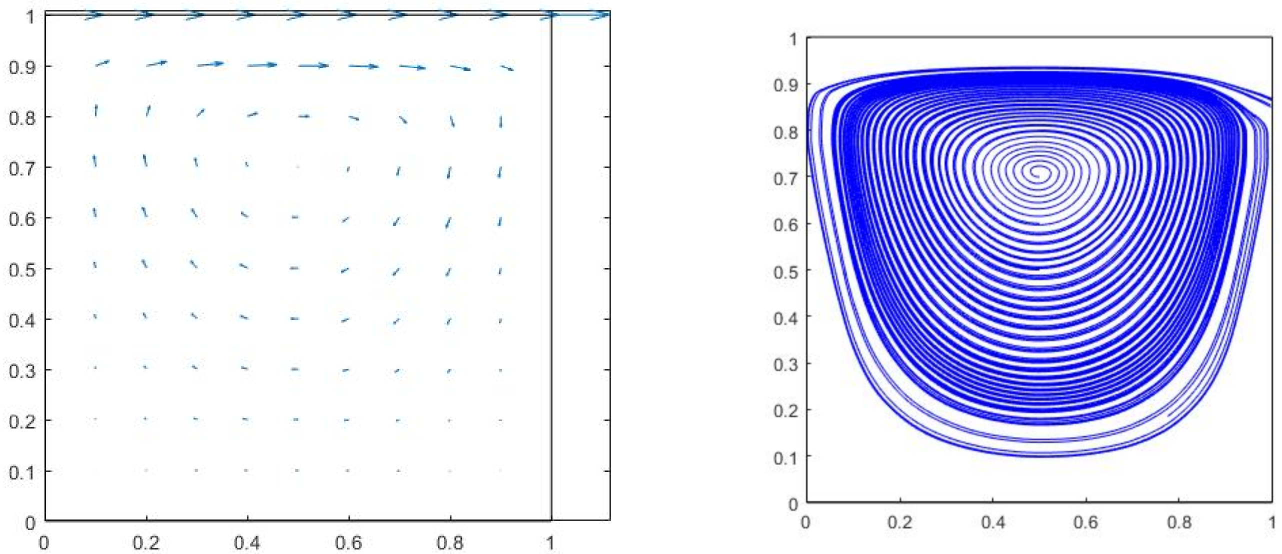

Figure 15. Velocity field represented by vectors (left) and streamlines (right) for P1P1 elements.

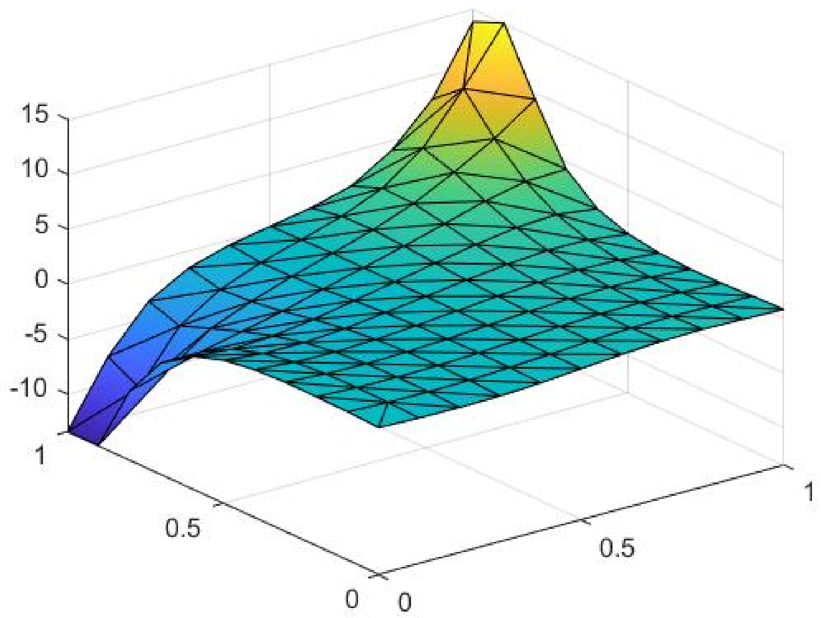

Figure 16. Pressure distribution over the domain.

From previous figures, we clearly see how instabilities, which where present in Q1Q1 and P1P1, have disappeared at the cost of smoothing the behaviour of the pressure distribution.

Problem Statement (Navier-Stokes)

Navier-Stokes equation

$$\begin{cases} -\nu \nabla^2 \vec{v} + (\vec{v}\cdot\nabla)\vec{v} + \nabla p = \vec{f} & \text{in } \Omega \\ \nabla\cdot\vec{v} = 0 & \text{in } \Omega \\ v = v_D & \text{on } \partial\Omega \end{cases}$$

Using Weighted residuals method:

$$\begin{cases} -\int \vec{w}\, \nu \nabla^2\vec{v}\, d\Omega + \int \vec{w}\cdot(\vec{v}\cdot\nabla)\vec{v}\, d\Omega + \int \vec{w}\cdot\nabla p\, d\Omega = \int \vec{w}\cdot\vec{f}\, d\Omega \\ \int q\, \nabla\cdot\vec{v}\, d\Omega \end{cases}$$

For Navier-Stokes equation we have an extra term which is the inertia term. So the weak form will remain the same with that extra term

$$\begin{cases} \int_\Omega \nabla \vec{w} : \nu \nabla \vec{v} \, d\Omega + \int \vec{w} \cdot (\vec{v} \cdot \nabla) \vec{v} \, d\Omega - \int p \nabla \cdot \vec{w} \, d\Omega = \int \vec{w} \cdot \vec{f} \\ \int q \nabla \cdot \vec{v} \, d\Omega = 0 \end{cases}$$

our system of equations will remain the same but now with an extra term which is non-linear with respect the velocity:

$$\begin{pmatrix} K + C(\vec{v}) & G^T \\ G & 0 \end{pmatrix} \begin{pmatrix} \vec{v} \\ p \end{pmatrix} = \begin{pmatrix} \vec{f} \\ 0 \end{pmatrix}$$

where $C(\vec{v})$ arises from $\int_\Omega \vec{w} \cdot (\vec{v} \cdot \nabla) \vec{v} \, d\Omega$

In this case, due to the non-linear term, the system of equations must be solved iteratively by finding the root using different methods, such as Picard or Newton-Rapson methods.

Results

When considering Re=100 and Q2Q1 elements, results obtained are:

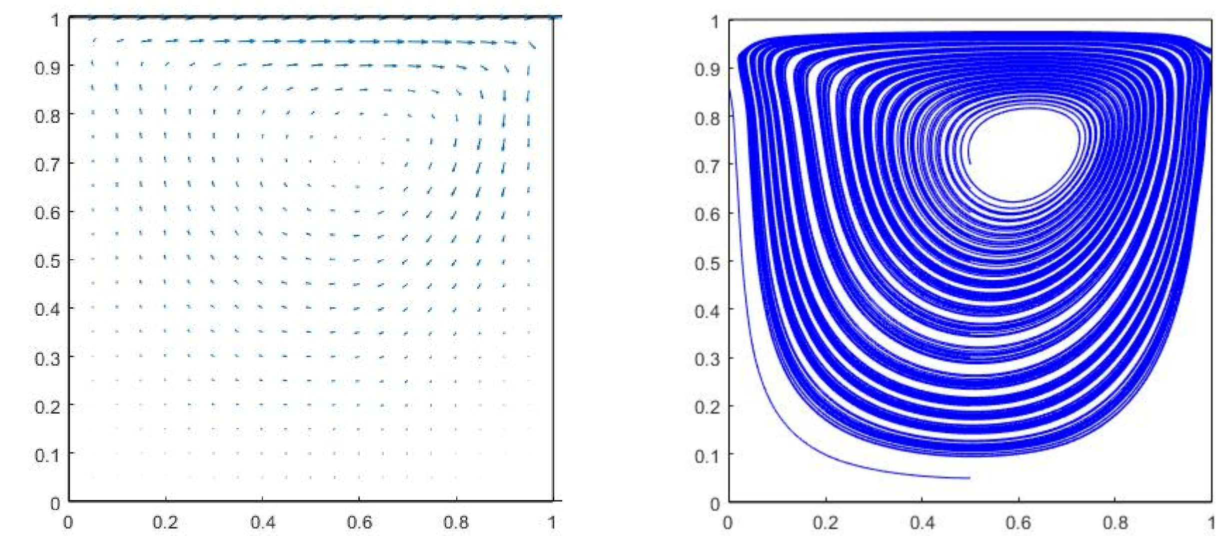

Figure 17. Velocity field represented by vectors (left) and streamlines (right) for Q2Q1 elements.

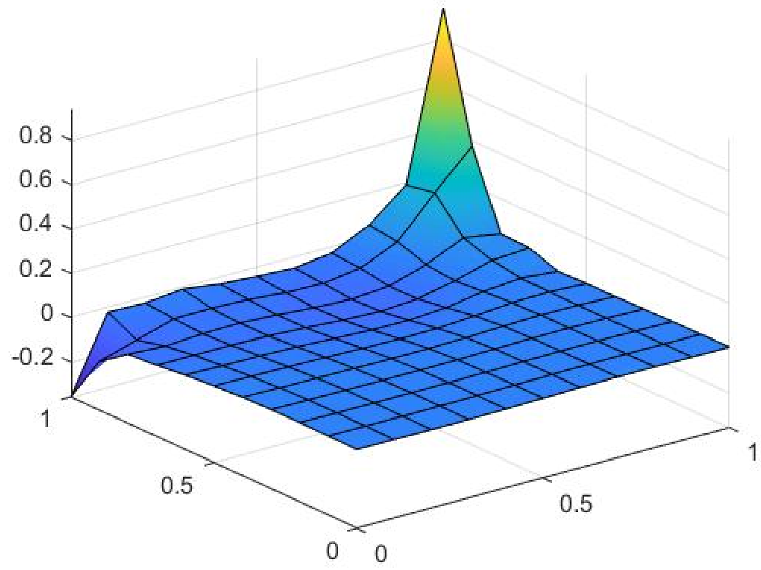

Figure 18. Pressure distribution over the domain.

Comparing the previous figures with those from Stokes equation, we see how velocity is no longer 'symmetrical' with respect to the lateral faces. This is due to the inertia term which moves the solution to the direction of the inner flow. Pressure also shows a difference in the two picks of the lateral corners between the inner flow and the lateral face.

An other consideration is that those results are the same for Picard and Newton-Rapson method since we are solving the same equation. The only difference is the iterative method being used. This is traduced to a different number of iterations needed to achieve the desired error, as it can be seen in *Figure 18.*

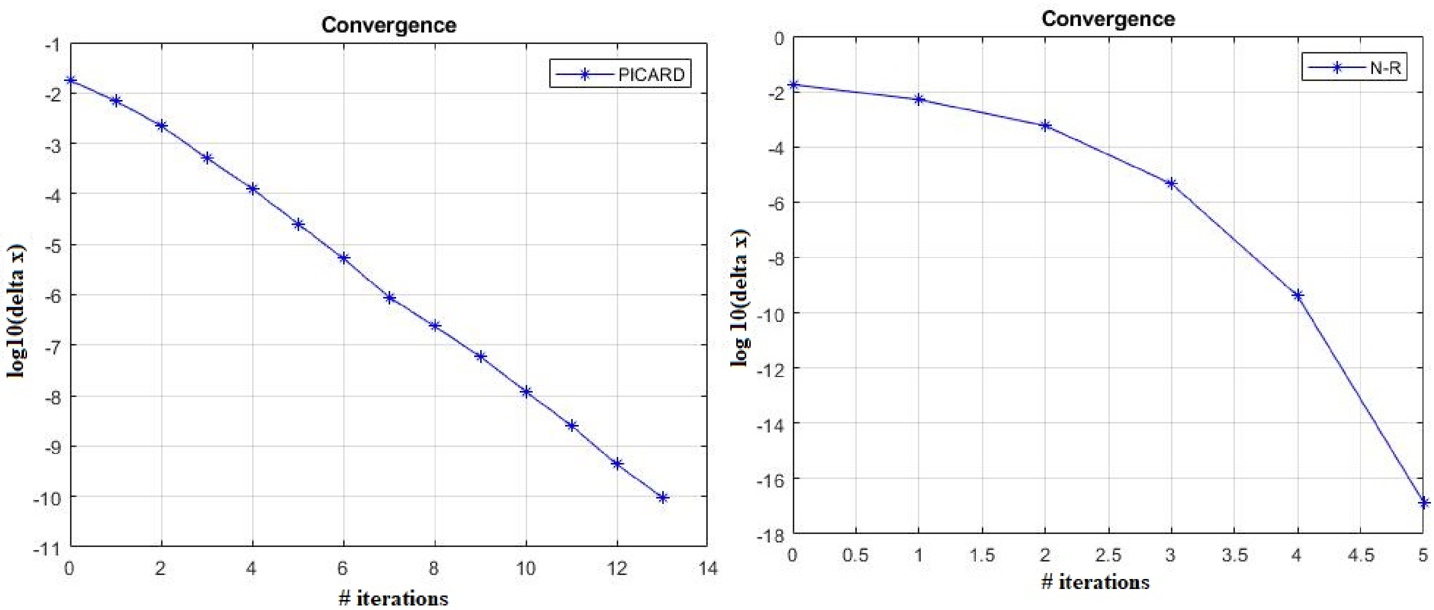

Figure 18. Logarithm of the error in terms of the number of iteration using Picard method (left) and Newton-Rapson method (right).

As it can be seen, Newton-Rapson method gets the solution faster than Picard, since the error is reduced quadratically for N-R and linearly for Picard method.

To finish this work, it must be said that the results for P2P1 have not been shown since they look the same than those of Q2Q1. Including figures which doesn’t present any difference with respect to the previous ones has no sense, so it has been decided not to include them.