HOMEWORK UNSTEADY CONVECTION

-FINITE ELEMENTS IN FLUIDS-

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For a transient advective, non-convective, non-reactive, system:

$u_t + a u_x = 0$ $x \in (0,1)$ $t \in (0,0.6)$
$u(x,0) = u_0(x)$ $x \in (0,1)$
$u(0,t) = 1$ $t \in (0,0.6)$

with initial condition:
$u_0(x) = 1$ *if* $x \leq 0.2$
$u_0(x) = 0$ *otherwise*

$a=1;$
$h = \Delta x = 2 \cdot 10^{-2}$
$\Delta t = 1{,}5 \cdot 10^{-2}$

Where $A \cdot \Delta u = (B u^n + f)$ is the LINEAR SYSTEM to solve.

- Compute Courant Number.
- Solve Problem using C-N in time and linear finite element for the Galerkin scheme in space. Is the solution accurate?
- Solve problem using 2nd order Lax-Wendroff method.
  Can we expect the solution to be accurate? If not, what changes are necessary?
- Solve problem with a 3d order Taylor-Galerkin method.

COURANT NUMBER

C=|a|*Δt/h=1*1,5*10^{-2}/2*10^{-2}=0,75

C^2=0,5625

- C-N (1D) unconditional stability
- TG2 (1D) stable if C^2<⅓→ **Possible instabilities!**
- TG3 (1D) stable If C^2<1

Defining the matrices: M, K, & C:

Me = Me + w_ig*(N_ig'*N_ig)
Ke = Ke + w_ig*(Nx_ig'*Nx_ig)
Ce = Ce + w_ig*(N_ig'*Nx_ig)

We choose problem type =4, which adequates program to initial conditions of this homework.

THETA METHOD+GALERKIN with C-N

Following a time stepping discretization, trucating 2nd order term $\frac{u(t^{n+1}) - u(t^n)}{\triangle t} = u_t(t^n)$.

and combining with advection formulae $u_t + (\boldsymbol{a} \cdot \boldsymbol{\nabla})u = s$,

we obtain:

$$\left(w, \frac{\triangle u}{\triangle t}\right) - \theta\left(\boldsymbol{\nabla} w, \boldsymbol{a}\,\triangle u\right) + \theta\left((\boldsymbol{a}\cdot\boldsymbol{n})w, \triangle u\right)_{\Gamma^{out}} = \left(\boldsymbol{\nabla} w, \boldsymbol{a}\,u^n\right) - \left((\boldsymbol{a}\cdot\boldsymbol{n})w, u^n\right)_{\Gamma^{out}} + \left(w, \theta h^{n+1} + (1-\theta)h^n\right)_{\Gamma_N^{in}} + \left(w, \theta s^{n+1} + (1-\theta)s^n\right)$$

recall: Galerkin W=N & Δu=N_j*Δu_j U=$N_j U_j$, developing a bit and considering s=0 we obtain:

A = M + 1/2*a*Δt*C;
B = -a*Δt*C;

To evaluate accuracy we compare exact and calculated solution at last step and compare each point the function 'u' calculated to the exact solution.

u_{calc}=u(:,nStep+1)'
X = u_{calc} - uu;
Dif=sum(abs(X))/nPt (SQUARED DIFFERENCES)

Mean squared error=**0.0751**

THETA METHOD+GALERKIN with C-N LUMPED MASS MATRIX

Analogously,

```
% Loop on Gauss points
  for ig = 1:ngaus
    N_ig = N(ig,:);
    Nx_ig = Nxi(ig,:)*2/h;
    w_ig = wgp(ig)*h/2;

    Me = Me + w_ig*(N_ig'*N_ig);
    Ke = Ke + w_ig*(Nx_ig'*Nx_ig);
    Ce = Ce + w_ig*(N_ig'*Nx_ig);

    MLe(1,1)=w_ig*N_ig(1);
    MLe(2,2)=w_ig*N_ig(2);
```

A = ML + 1/2*a*Δt*C;
B = -a*Δt*C;

Now the Mean squared error is **0.2677,** some accuracy has been lost with the approximation of M to M_L.

TG2 (OR L-W) +GALERKIN (2ND ORDER)

Following a time stepping discretization, trucating 3nd order terms,

$$\frac{u(t^{n+1})-u(t^n)}{\triangle t} = u_t(t^n) + \frac{1}{2}\triangle t\, u_{tt}(t^n)$$

and combining with advection formulae $u_t + (\boldsymbol{a}\cdot\boldsymbol{\nabla})u = s$,

we obtain:

$$\begin{aligned}\left(w,\frac{\triangle u}{\triangle t}\right) = \Big(\boldsymbol{a}\cdot\boldsymbol{\nabla} w, u^n + \frac{\triangle t}{2}[s^n - (\boldsymbol{a}\cdot\boldsymbol{\nabla})u^n]\Big)\\ - \Big((\boldsymbol{a}\cdot\boldsymbol{n})w, u^n + \frac{\triangle t}{2}[s^n - (\boldsymbol{a}\cdot\boldsymbol{\nabla})u^n]\Big)_{\Gamma^{out}}\\ + \big(w, h^{n+1/2}\big)_{\Gamma_N^{in}} + \Big(w, s^n + \frac{\triangle t}{2}s_t^n\Big)\end{aligned}$$

<u>recall</u>: Galerkin W=N & $\Delta u = N_j{}^*\Delta u_j$ $U=N_jU_j$, developing a bit and considering s=0 we obtain:

A = M;
B = a*C*Δt+K*Δt²*a²/2;

Mean squared error makes no sense, it's $\mathbf{10^{23}}$.

C^2=0,5625 , so it's bigger than ⅓. In order to make it a stable method whe should change Courant number, diminish it. ***This could be done by decreasing Δt (e.g.1,5*10^{-2}-->1*10^{-2}) or increasing h (e.g. 2*10^{-2}-->4*10^{-2}).*** For example doing the latter, the number of elements is 25, -the half- , and the error is lower: 10^9

TG2 (OR L-W) +GALERKIN (2ND ORDER) LUMPED MASS MATRIX

Recall:

```
MLe(1,1)=w_ig*N_ig(1);
MLe(2,2)=w_ig*N_ig(2);
```

A = ML;
B = a*C*Δt+K*Δt²*a²/2;

Mean squared error makes no sense, $\mathbf{10^{18}}$.
Again**,** In order to make it a stable method whe should change Courant number, diminish it.

TG3 + GALERKIN (3D ORDER)

Following a time stepping discretization, trucating 4th order terms,

$$\frac{u(t^{n+1}) - u(t^n)}{\triangle t} = u_t(t^n) + \frac{1}{2}\triangle t\, u_{tt}(t^n) + \frac{1}{6}\triangle t^2 u_{ttt}(t^n)$$

and combining with advection formulae $u_t + (\boldsymbol{a}\cdot\boldsymbol{\nabla})u = s$,

we obtain:

$$\begin{aligned}(w, \frac{\triangle u}{\triangle t}) &+ \boxed{\frac{\triangle t^2}{6}(\boldsymbol{a}\cdot\boldsymbol{\nabla} w, \boldsymbol{a}\cdot\boldsymbol{\nabla}\frac{\triangle u}{\triangle t})} - \boxed{\frac{\triangle t^2}{6}((\boldsymbol{a}\cdot\boldsymbol{n})w, \boldsymbol{a}\cdot\boldsymbol{\nabla}\frac{\triangle u}{\triangle t})_{\Gamma^{out}}} \\ &= (\boldsymbol{a}\cdot\boldsymbol{\nabla} w, u^n - \frac{\triangle t}{2}(\boldsymbol{a}\cdot\boldsymbol{\nabla} u^n)) - ((\boldsymbol{a}\cdot\boldsymbol{n})w, u^n - \frac{\triangle t}{2}(\boldsymbol{a}\cdot\boldsymbol{\nabla} u^n))_{\Gamma^{out}} \\ &+ \frac{\triangle t}{2}(\boldsymbol{a}\cdot\boldsymbol{\nabla} w, s^{n+1/3}) - \frac{\triangle t}{2}((\boldsymbol{a}\cdot\boldsymbol{n})w, s^{n+1/3}) \\ &+ (w, \frac{3}{4}s^{n+2/3} + \frac{1}{4}s^n) + (w, \frac{3}{4}h^{n+2/3} + \frac{1}{4}h^n)_{\Gamma_N^{in}}\end{aligned}$$

recall: Galerkin W=N & $\Delta u=N_j*\Delta u_j$ $U=N_jU_j$, developing a bit and considering s=0 we obtain:

$$\mathbf{A = M + \Delta t^{2}*a^{2}*K/6;}$$
$$\mathbf{B = a*C*\Delta t-\Delta t^{2}*a^{2}*K/2;}$$

Mean squared error is **0.7672**.