ADNAN ALI IMRAN

HW-3 FEF

$$u_t^\epsilon + u^\epsilon u_x^\epsilon = \epsilon u_{xx}^\epsilon \qquad \text{For } (x,t) \in [-1,1] \times [0,T]$$

$$u^\epsilon(x,0) = u_0(x) \qquad \text{For } x \in [-1,1]$$

$$u^\epsilon(-1,t) = u^\epsilon(1,t) = 0 \qquad \text{For } t \in [0,T]$$

One Step T-Galerkin

Sol

let take 2nd-order time expantion

$$u^{m+1} = u^m + \Delta t\, u_t^m + \frac{1}{2} \Delta t^2 u_{tt}^m \quad — \text{①}$$

$$u_t = -f_x(u) + \epsilon u_{xx}$$

$$u_{tt} = -f_{xt}(u) + \epsilon u_{xxt} \qquad \text{where} \qquad \epsilon u_{xxt} = -f_x u_t + \epsilon[-f_x + \epsilon U_{xx}]_{xx}$$

$\epsilon (u_t)_{xx}$

$$\because \epsilon U_{xxt} = -f_x u_t + \epsilon[-f_x + \epsilon U_{xx}]_{xx}$$

So,
$$u_{tt} = -f_x u_t + \epsilon[-f_{xxx} + \epsilon u_{xxxx}]$$
$$= -(f_u U(t))_x + \epsilon[-f_{xxx} + \epsilon U_{xxxx}]$$

So, As we have Eq ①

$$\frac{U^{m+1} - U^m}{\Delta t} = U_t^m + \frac{1}{2} \Delta t\, U_{tt}^m \quad — \text{②}$$

Substitute u_t & u_{tt} values in Eq-②

$$\frac{U^{m+1} - U^m}{\Delta t} = (-f_x + \epsilon U_{xx}) + \frac{1}{2} \Delta t\left(-(f_u U_t)_x + \epsilon[-f_{xxx} + \epsilon U_{xxxx}]\right)$$

Integral Form

$$\int w \left(\frac{U^{m+1} - U^m}{\Delta t}\right) dx = \int w(-f_x + \epsilon U_{xx})\, dx + \int \frac{1}{2} \Delta t \cdot w\left(-(f_u U_t)_x + \epsilon[-f_{xxx} + \epsilon U_{xxxx}]\right)$$

As we know

$$uv = \int u\,dv + \int v\,du \implies \int u\,dv = uv - \int v\,du$$

Same,

$$\int w f_x\,dx = wf - \int f w_x\,dx$$

And

$$\int w \epsilon U_{xx}\,dx = \epsilon w U_x - \int \epsilon U_x w_x\,dx$$

And

$$\int w\left[f_u(-f_{xx} + \epsilon U_{xxx})\right]dx = w f_u(-f_x + \epsilon U_{xx}) - \int f_u(-f_x + \epsilon U_{xx}) w_x\,dx$$

And

$$\int w(-f_{xxx} + \epsilon U_{xxxx})dx = w(-f_{xx} + \epsilon U_{xxx}) - \int(-f_{xx} + \epsilon U_{xxx}) w_x\,dx$$

So,

$$\int w\left(\frac{U^{n+1} - U^n}{\Delta t}\right)dx = wf - \int f w_x\,dx + \epsilon w U_x - \int w_x \epsilon U_x\,dx$$
$$+ \frac{1}{2}\Delta t\left(-\left[w f_u(-f_x + \epsilon U_{xx}) - \int f_u(-f_x + \epsilon U_{xx}) w_x\,dx\right)\right.$$
$$\left. + \epsilon w(-f_{xx} + \epsilon U_{xxx}) - \epsilon \int\left[-f_{xx} + \epsilon U_{xxx}\right] w_x\,dx\right)$$

Weak Form

2-Step T-Galerkin

let's consider

$$U^{n+\frac{1}{2}} = U^n + \frac{\Delta t}{2} U_t^n \quad \text{— (i)}$$

$$U^{n+1} = U^n + \Delta t\, U_t^{n+1/2} \quad \text{— (ii)}$$

And

$$U_t^n = -f_x + \epsilon U_{xx}$$

(i) $$U^{m+\frac{1}{2}} = U^m + \frac{\Delta t}{2}\left(\epsilon U_{xx}^m - f_x(U^m)\right)$$

(ii) $$U^{m+1} = U^m + \Delta t\left(\epsilon U_{xx}^{m+1/2} - f_x\left(U^{m+1/2}\right)\right)$$

And

$$\frac{U^{m+1} - U^m}{\Delta t} = \epsilon U_{xx}^{m+1/2} - f_x\left(U^{m+1/2}\right)$$

$\boxed{\text{Weak Form}}$

$$\int w\left(\frac{U^{m+1} - U^m}{\Delta t}\right)dx = -\int\left(w_x\,\epsilon U_x^{m+1/2}\right)dx + \int_{\Gamma_{out}} w_x\,\epsilon U_x^{m+1/2}$$

$$+ \int w_x\, f(U^{m+1/2})\,dx - \int_{\Gamma_{out}} w_x\, f(U^{m+1/2})\,dx$$