Finite Element in Fluids

Un-Steady Convection

Home Work -4

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MS-Computational Mechanics

Problem No.1

Problem data

$$\begin{cases} u_t + au_x = 0 & x \in (0,1),\ t \in (0, 0.6] \\ u(x,0) = u_0(x) & x \in (0,1) \\ u(0,t) = 0 & t \in (0, 0.6] \end{cases}$$

$$u_0(x) = \begin{cases} \frac{1}{2}(1 + \cos(\pi(x - x_0)/\sigma)) & \text{if } |x - x_0| \le \sigma, \\ 0 & \text{otherwise} \end{cases}$$

$$a = 1, x_0 = 0.2, \sigma = 0.12, \Delta x = 2 \cdot 10^{-2}$$

Keeping in view the problem data we have selected the **Time Steps (n =50)** & **Space Steps (I = 50)** in such a way to have a value of **Courant No** equal to **0.6**.

Matrixes involved in the weak formulation of different methods.

Mass matrix M = $\int N_a N_b$ dΩ

Convection matrix C = $\int N_a(\text{a}.\nabla N_b)$dΩ

Boundary Conditions B^{out} = $\int N_a N_b (a.n)$dΓ

K Matrix K = $\int \nabla N_a (\nabla N_b)$dΩ

The problem is solved by 4 different methods to evaluate the best approximate method. All four methods are evaluated against stability and accuracy. The result are presented below and each method is discussed with its advantages and drawbacks..

a. Lax Wendroff + Galerkin Method (LW)

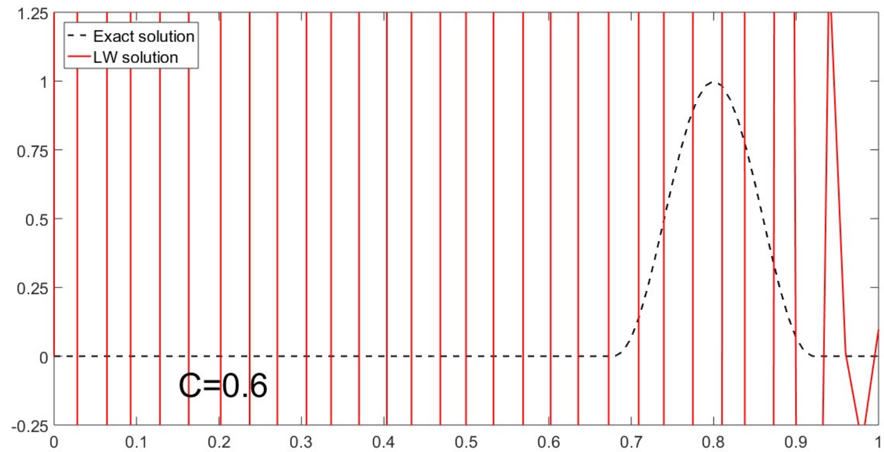

Figure-1: Response in Lax Wandroff + Galerkin scheme with Courant No. 0.6

Comment:

This method is stable up to (Courant No) $C^2 \leq 1/3$.The system has a constant matrix and consistent mass matrix. The method exploded at C = 0.6 as C > 0.57 as it can be seen in figure 1.

b. Lax Wendroff with lumped Mass + Galerkin Method (LW-FD)

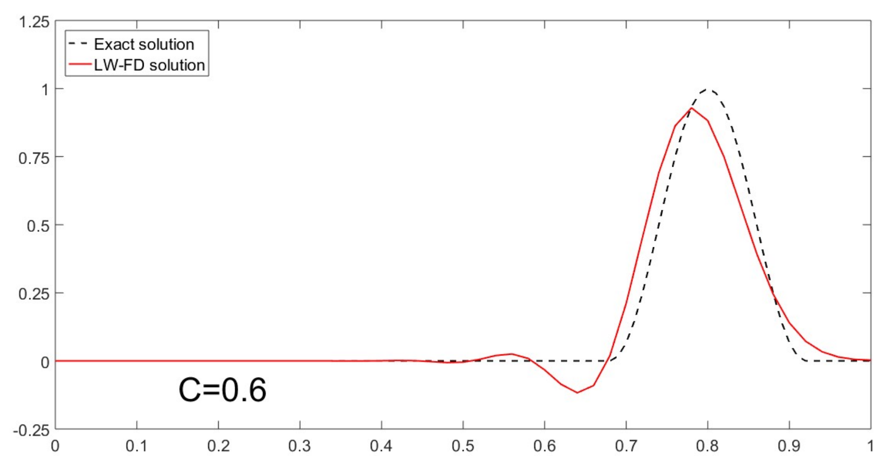

Figure-2: Response in Lax Wandroff with lumped mass + Galerkin scheme with Courant No. 0.6

Comment:

The finite element scheme corresponding to diagonal mass instead of consistent mass matrix is called Lax Windroff with lumped mass method. It is for more economical from a computational point of view and also have larger domain of stability. This method has stabilized the TG-2 scheme by introducing the lumped mass. The solution is stable at C = 0.6, as shown in figure 2.

c. Crank-Necolson + Galerkin Method (CN)

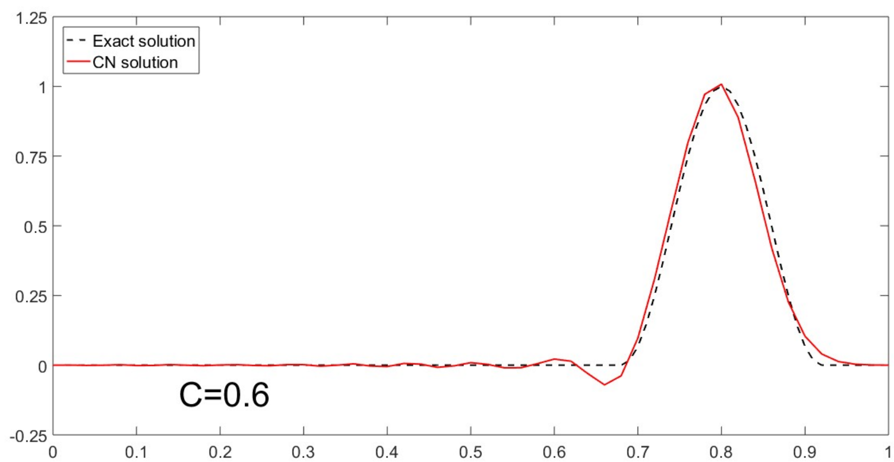

Figure-3: Response in Crank-Necolson scheme with Courant No. 0.6

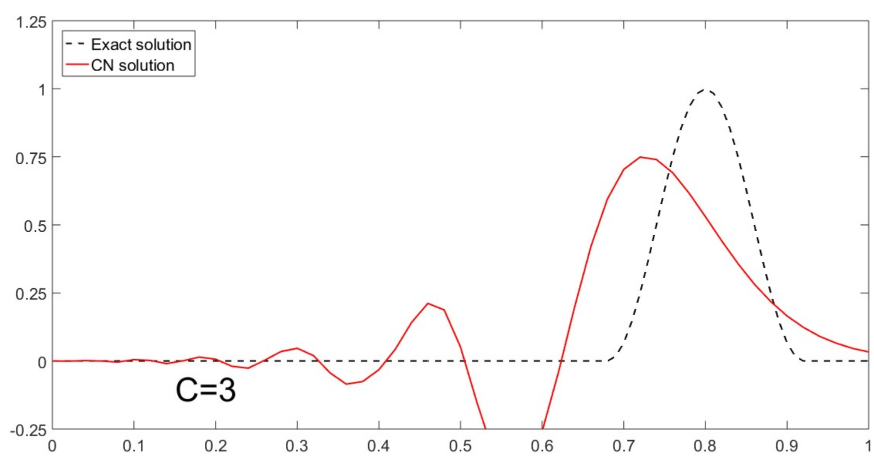

Figure-4: Response in Crank-Necolson scheme with Courant No. 3.0

Comment:

This method is un-conditionally stable, even at higher values of Courant No say C = 3, as shown in figure 4, the solution is still stable but behave badly in terms of accuracy but it never explodes.

d. Crank-Necolson with lumped mass Method (CN-FD)

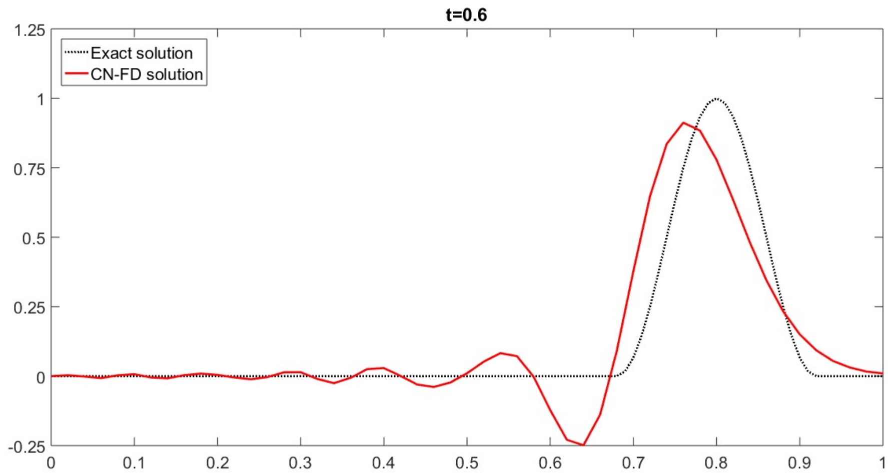

Figure-5: Response in Crank-Necolson with lumped mass scheme with Courant No. 0.6

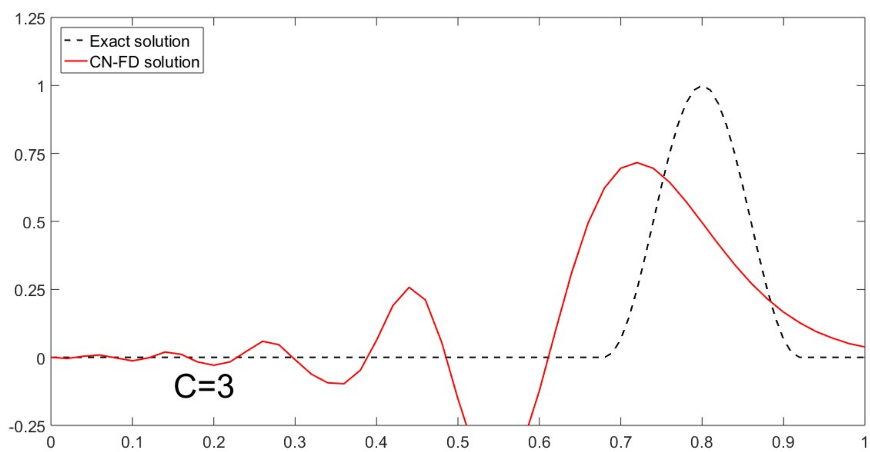

Figure-6: Response in Crank-Necolson with lumped mass scheme with Courant No. 3.0

Comment:

Crank Nesolcon with finite element difference scheme is also un conditionally stable but with lesser accuracy as compared to Crank Nesolcon scheme without lumped mass. The solution shown in figures 5 & 6, it is clearly evident that it is less accurate.

Problem No.2

Problem data

$$\begin{cases} u_t + a u_x = 0 & x \in (0,1),\ t \in (0, 0.6] \\ u(x,0) = u_0(x) & x \in (0,1) \\ u(0,t) = 1 & t \in (0, 0.6] \end{cases}$$

$$u_0(x) = \begin{cases} 1 & \text{if } x \leq 0.2, \\ 0 & \text{otherwise} \end{cases}$$

$$a = 1, \Delta x = 2 \cdot 10^{-2}, \Delta t = 1.5 \cdot 10^{-2}$$

Courant No:

Courant No. C = IaI*(dt/h), so with given data,

C = 0.75

Matrixes involved in the weak formulation of different methods.

Mass matrix M = $\int N_a N_b$ dΩ

Convection matrix C = $\int N_a(\text{a}.\nabla N_b)$dΩ

Boundary Conditions B^{out} = $\int N_a N_b (a.n)$dΓ

K Matrix K = $\int \nabla N_a(\nabla N_b)$dΩ

The problem is solved by Crank-Necolson + Galerkin Method, Lax-Wendroff + Galerkin (TG 2nd Order) Method and Taylor + Galerkin (TG 3rd Order) Method and results with comments are presented below.

a. Crank-Necolson + Galerkin Method (CN)

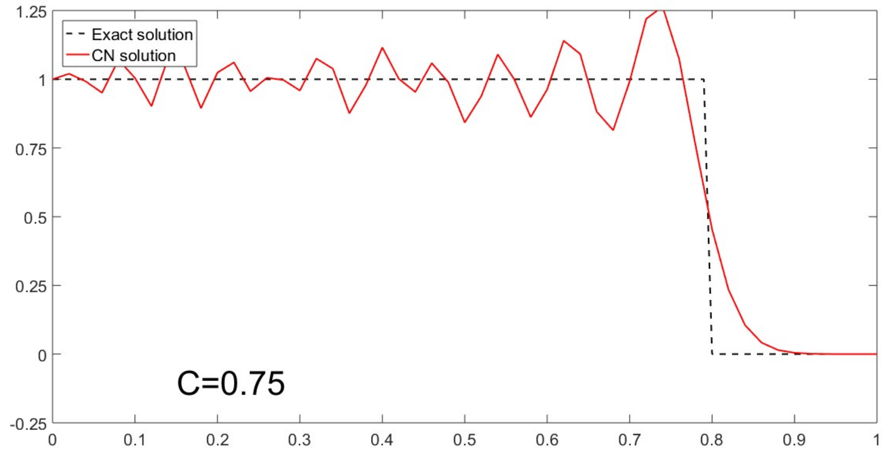

Figure-7: Response in Crank-Necolson with Courant No. 0.75

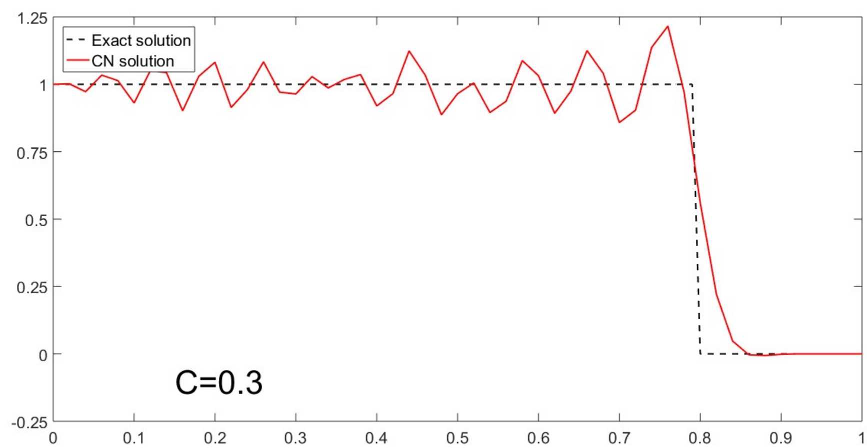

Figure-8: Response in Crank-Necolson with Courant No. 0.3

Comment:

In this particular case the solution is not so accurate but it is stable because Crank Nesolcon scheme is un-conditionally stable regardless the values of Courant Number but accuracy can be affected with higher values of C.

b. Lax-Wendroff + Galerkin Method (TG-2)

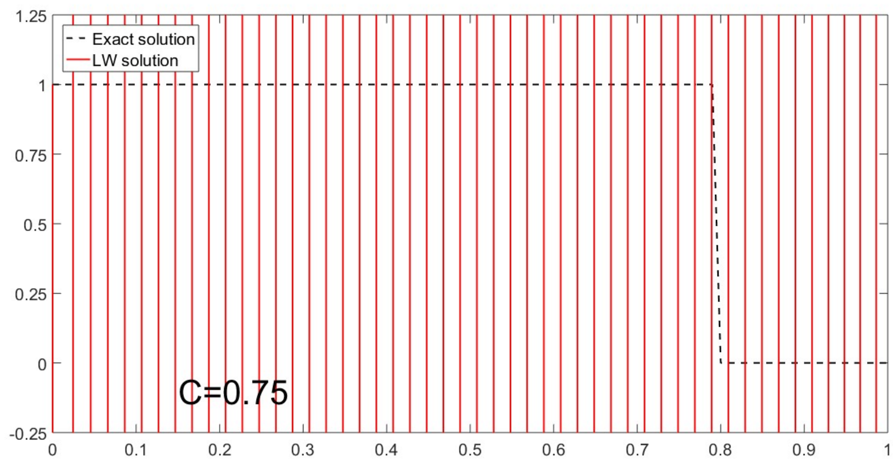

Figure-9: Response in Lax-Wendroff (TG-2) with Courant No. 0.75

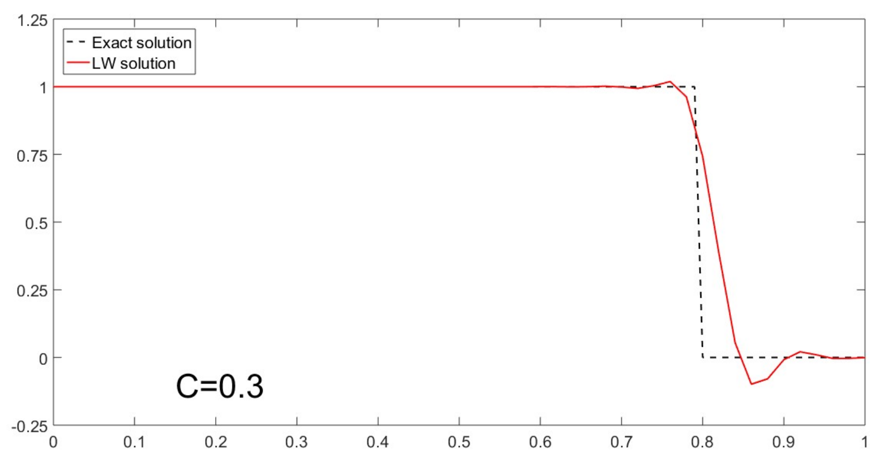

Figure-10: Response in Lax-Wendroff (TG-2) with Courant No. 0.3

Comment:

This method is stable up to (Courant No) $C^2 \leq 1/3$ or $C \leq 0.57$ that's why solution exploded at C = 0.75 (as shown in figure-9). To have a stable solution, we've to decrease the value of Courant Number. Figure-10 shows the stable solution at C = 0.3.

c. Taylor + Galerkin (3rd Order) Method (TG-3)

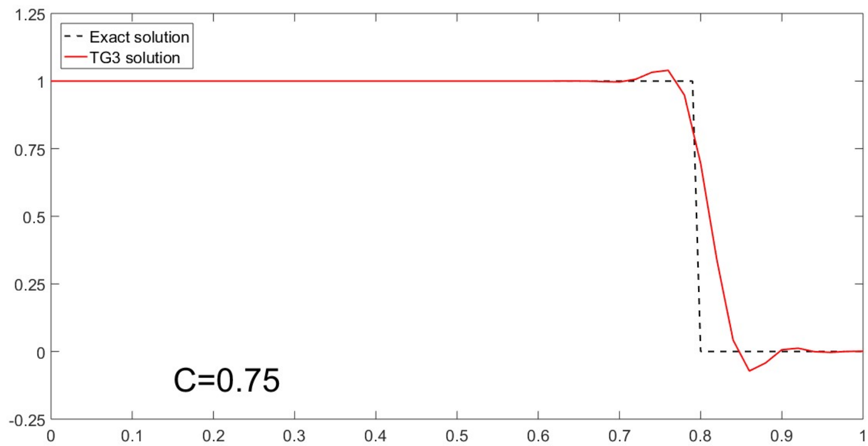

Figure-11: Response in Taylor + Galerkin (3rd Order) with Courant No. 0.75

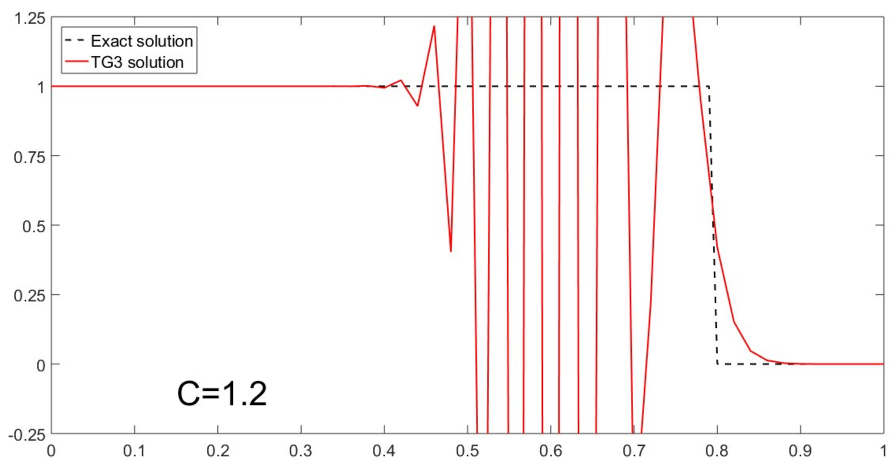

Figure-12: Response in Taylor + Galerkin (3rd Order) with Courant No. 1.2

Comment:

Taylor-Galerkin 3rd order scheme is stable as long $C^2 \leq 1$. The solution is stable at C = 0.75 as shown in figure-11. As C crosses the stability limit, the solution exploded. As shown in figure-12.