Finite Element in Fluids

Incompressible Flow

Navier Stokes Equation

Picard + Newton Raphson Method

Home Work -8

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MS-Computational Mechanics

Problem

Steady NS Equation

$$\begin{aligned} -\nu\nabla^2\boldsymbol{v} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v} + \boldsymbol{\nabla}p &= \boldsymbol{b} && \text{in } \Omega, \\ \boldsymbol{\nabla}\cdot\boldsymbol{v} &= 0 && \text{in } \Omega, \\ \boldsymbol{v} &= \boldsymbol{v}_D && \text{on } \Gamma_D, \\ \boldsymbol{n}\cdot\boldsymbol{\sigma} &= \boldsymbol{t} && \text{on } \Gamma_N, \end{aligned}$$

The Galerkin formulation is derived by substituting a linear combination of basic functions for velocity and pressure and the complete set of basic functions for the test functions. Since the convective terms are non-linear it is necessary to use some linearization scheme. Commonly used schemes are the Picard iteration where the convective terms at the new level are approximated by:

$$\mathbf{u}^{k+1}\cdot\nabla\mathbf{u}^{k+1} \approx \mathbf{u}^{k}\cdot\nabla\mathbf{u}^{k+1},$$

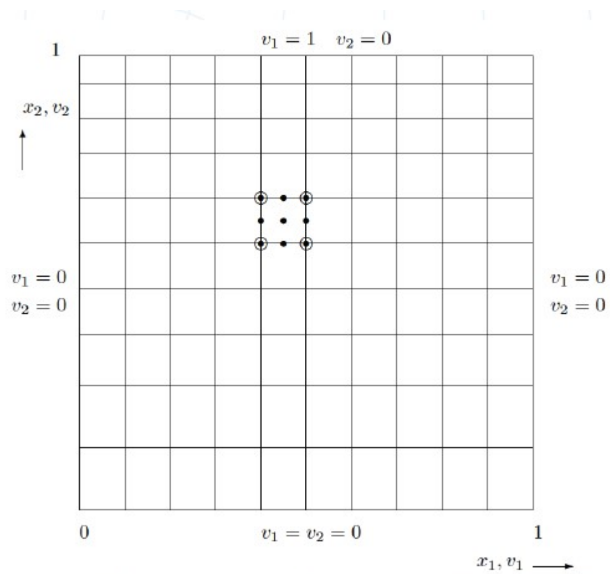
And the Newton scheme

$$\mathbf{u}^{k+1}\cdot\nabla\mathbf{u}^{k+1} \approx \mathbf{u}^{k}\cdot\nabla\mathbf{u}^{k+1} + \mathbf{u}^{k+1}\cdot\nabla\mathbf{u}^{k} - \mathbf{u}^{k}\cdot\nabla\mathbf{u}^{k}.$$

k denotes the old iteration level and *k + 1* the new.

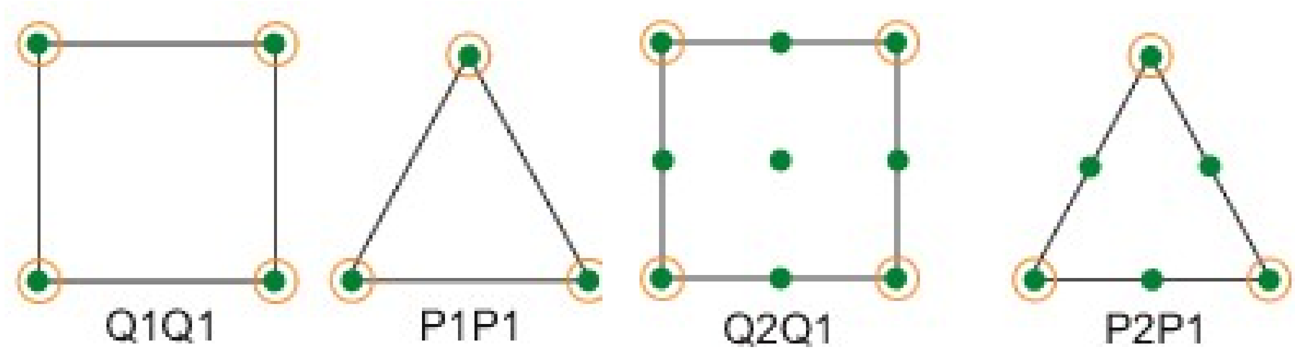
Picard Method

The matlab code is modified to implement Picard method for given problem and simulation results are presented below.

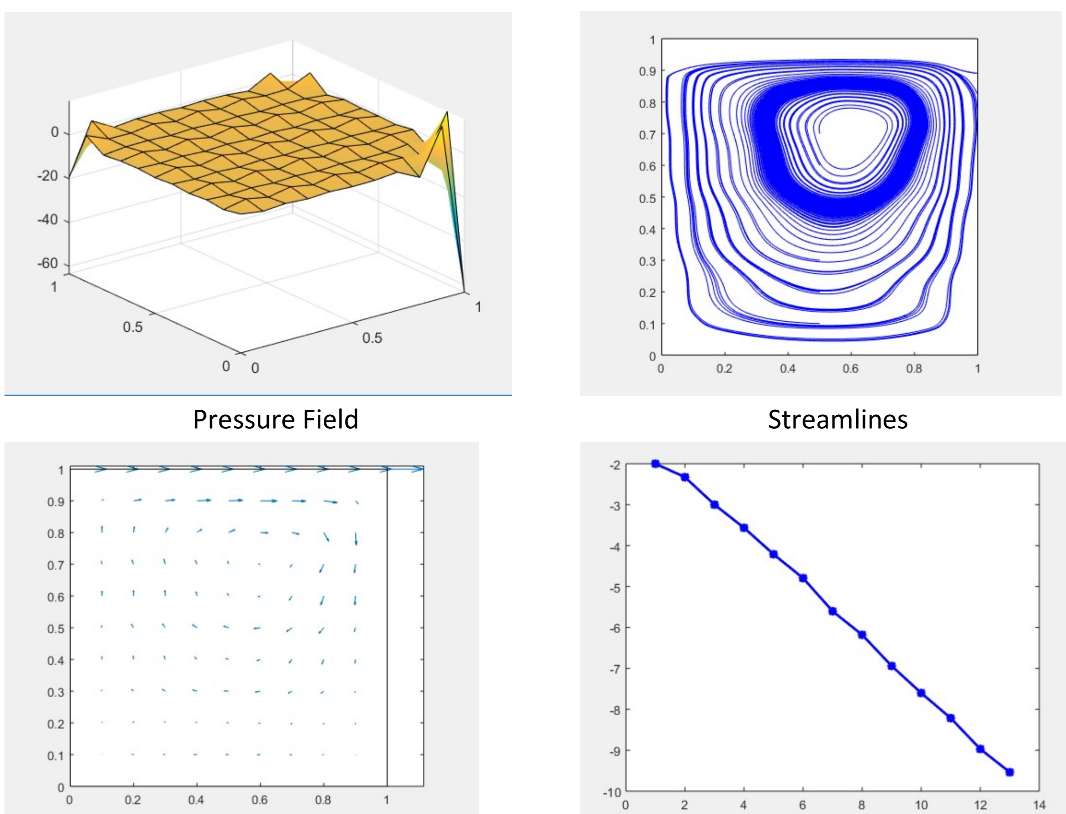


For Re =100, (Reynold Number) The two dimensional problem in the square domain Ω =]0, 1[x]0, 1[, with boundary conditions can be seen in above diagram. It poses a close solution with the velocity field v = (v1,v2) and pressure field p.

The problem is descritized with 10 elements in each direction. Four different types of elements have been chosen to analysis the problem. Q1Q1, P1P1, Q2Q1 & P2P1.



a) Q1Q1 Element



Pressure Field

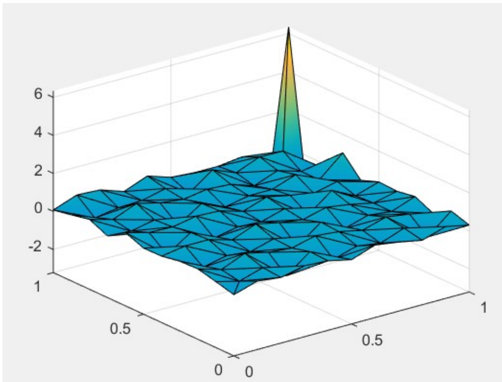
Streamlines

Elements scheme

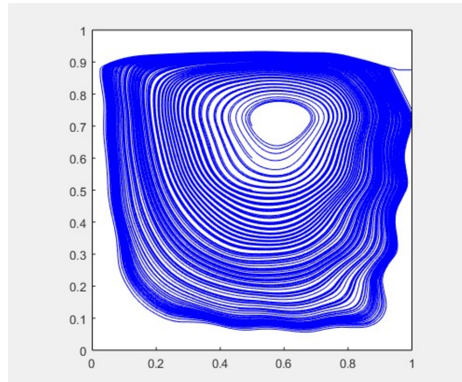
Velocity vectors

Figure-1: Response of Q1Q1 element with Picard Method

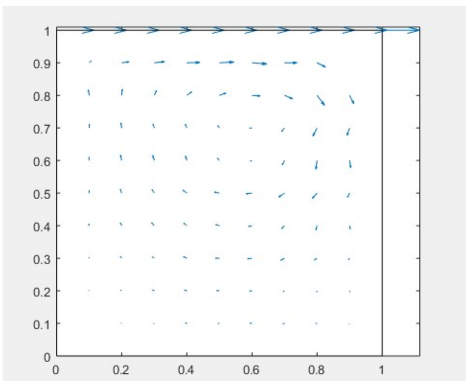
b) P1P1 Element



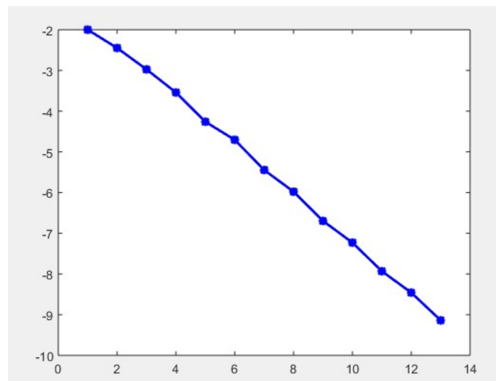
Pressure Field



Streamlines



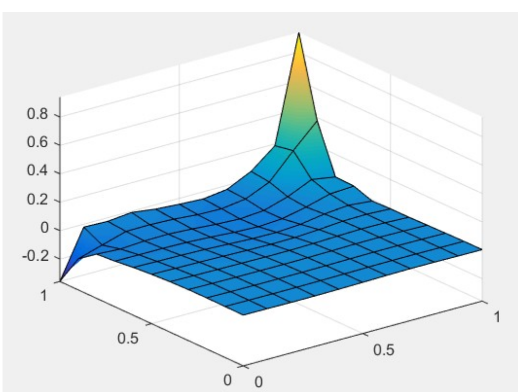
Elements scheme



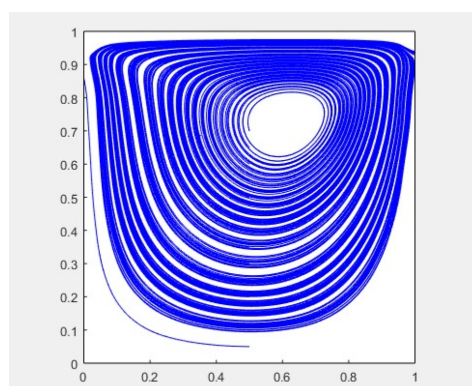
Velocity vectors

Figure-2: Response of P1P1 element with Picard Method

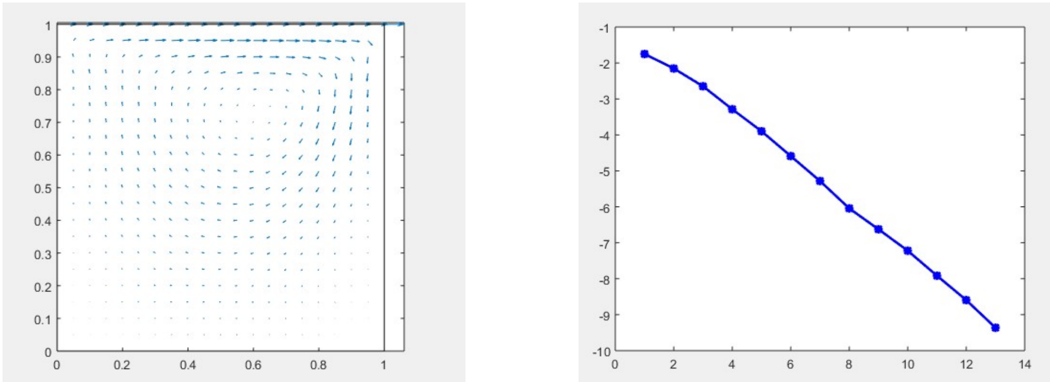
c) Q2Q1 Element



Pressure Field



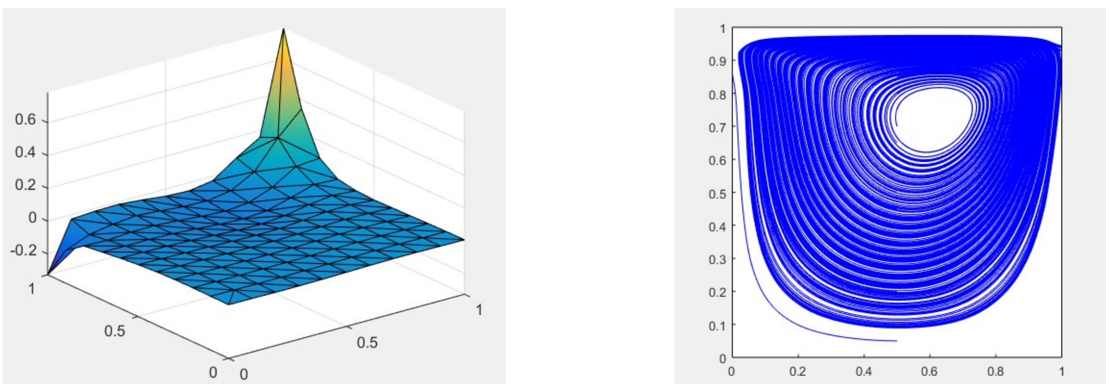
Streamlines



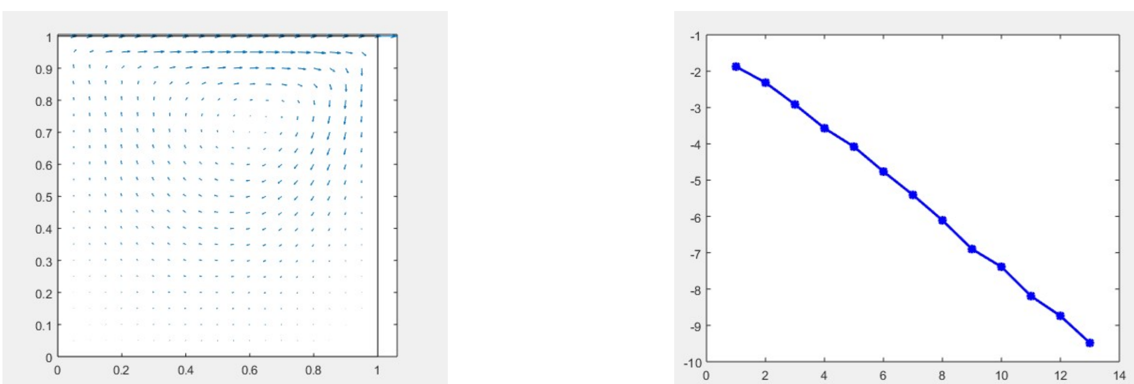
Elements scheme　　　　Velocity vectors

Figure-3: Response of Q2Q1 element with Picard Method

d) P2P1 Element



Pressure Field　　　　Streamlines



Elements scheme　　　　Velocity vectors

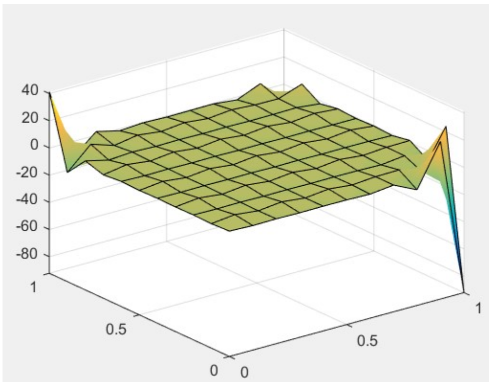
Figure-4: Response of P2P1 element with Picard Method

Comments:

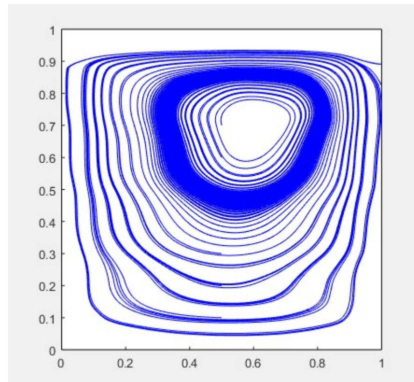
The convergence is slow and linear with Picard method and its more robust with lesser inner iterations. It can be seen that with Picard method the solution is converged with total of 13 iterations at Re =100 and both Q2Q1 & P2P1 elements give stable finite solutions. While Q1Q1 & P1P1 do not give stable due to fact that they do not satisfy LBB condition.

Newton Raphson Method

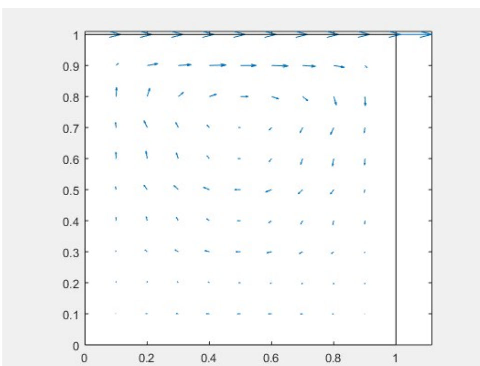
e) Q1Q1 Element



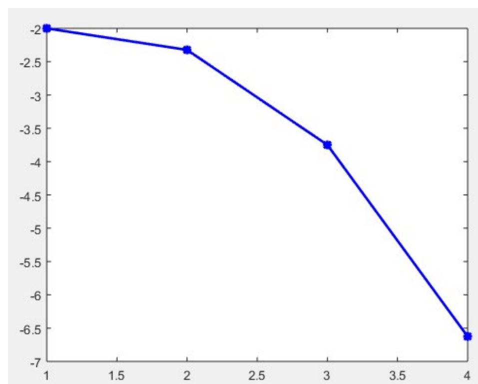
Pressure Field



Streamlines



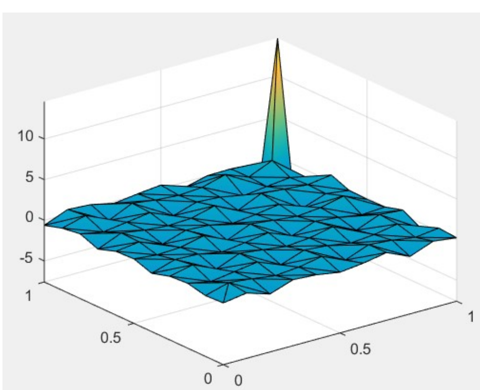
Elements scheme



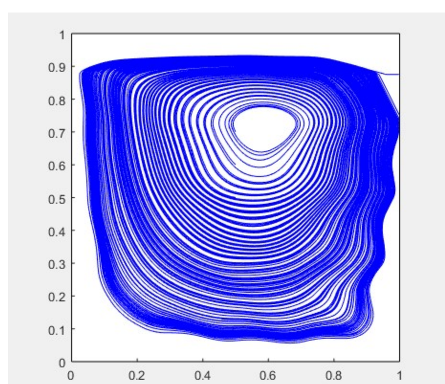
Velocity vectors

Figure-5: Response of stabilized Q1Q1 element with NR Method

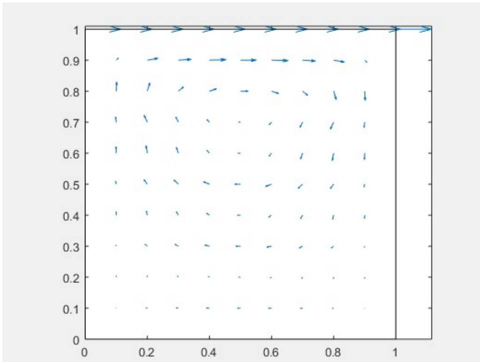
f) P1P1 Element

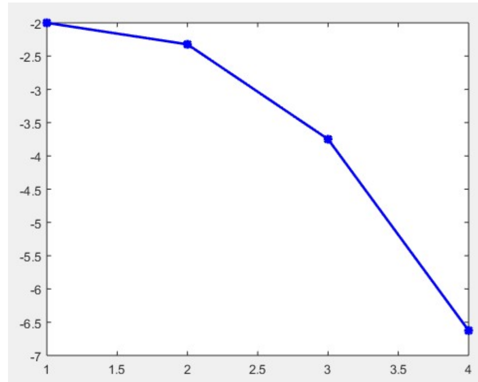


Pressure Field



Streamlines

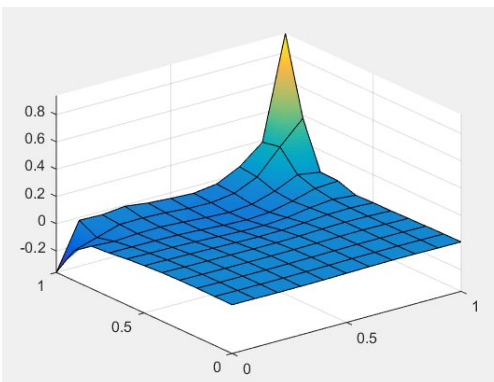


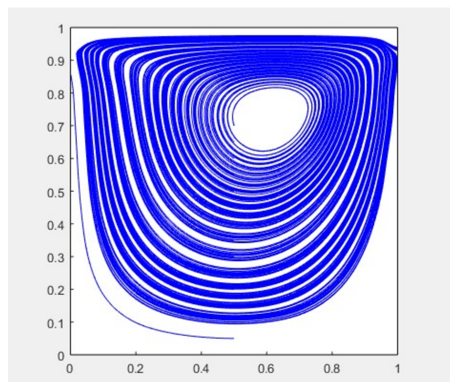


Elements scheme Velocity vectors

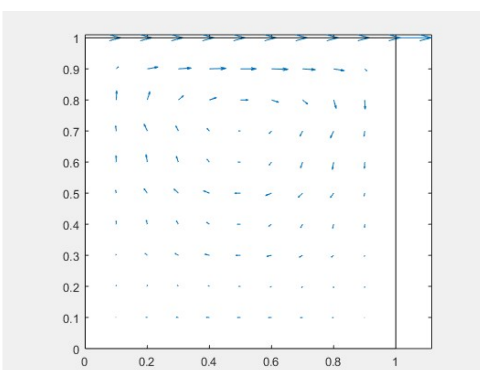
Figure-6: Response of stabilized P1P1 element with NR method

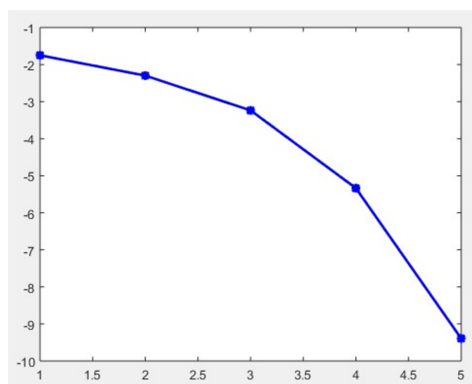
g) Q2Q1 Element





Pressure Field Streamlines

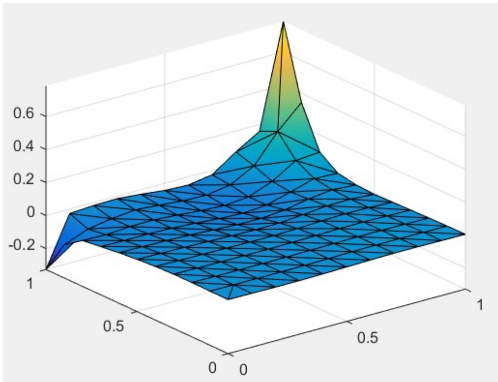




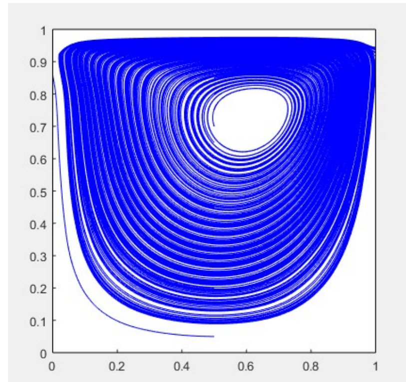
Elements scheme Velocity vectors

Figure-7: Response of stabilized Q2Q1 element with NR method

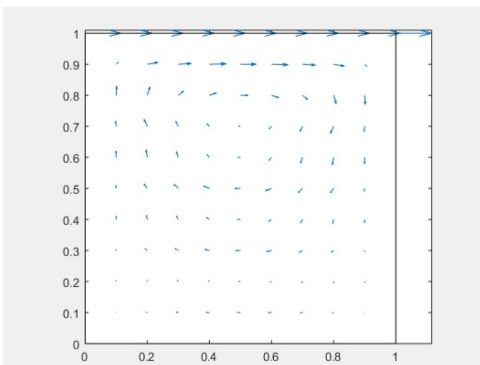
h) P2P1 Element



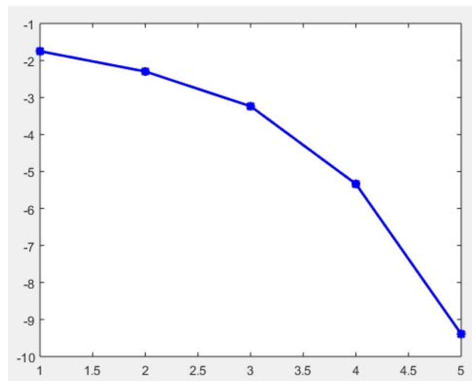
Pressure Field



Streamlines



Elements scheme



Velocity vectors

Figure-8: Response of stabilized P1P1 element with NR method

Comments

It is evident that NR method is faster method as compared to Picard method because with NR method the solution is converged in 5 steps and even in 4 steps for non stable Q1Q1 & P1P1 elements.
The convergence rate is linear in case of Picard method while the convergence rate is quadratic with Newton Raphson method.
NR gives faster convergence while it involves more inner iterations.