FEF Assignment 1

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1 Introduction

Finite Element Method is by far the most famous and used numerical method to solve partial differential equations. It is highly developed for solving solid and structural mechanics problems. However, the classical Galerkin Finite element method fails to provide good results when it comes to solve flow problems. That is due to several challenges in the differential equations governing flow problems. One of them is the steady transport problem which includes a convection term which is a source of problem if it is more dominant than the diffusive term. The Galerkin method fails to provide its famous symmetry that leads to minimizing the error in the energy norm leading to unstable solution. Therefore, several methods are used to provide better results such as: early Petrov-Galerkin methods, Streamline Upwind Petrov-Galerkin (SUPG) method, and Galerkin Least-Squared (GLS) method.

This report starts with introducing the 1d differential equation to be solved and its weak form. Then, it introduces the modification using SUPG and GLS methods. Modifications are made to the source term and quadratic elements are then used and the basic results are shown and discussed.

2 Strong, weak form, SUPG, and GLS stabilization methods

The simplified differential equation with given boundary conditions that is been solved is given as:

$$au_x - \nu u_{xx} = f \quad in\ ]0,1[$$
$$u(0) = 0, \qquad u(1) = 1$$

where u is the transported quantity, a is the convection velocity (a=1), ν is the coefficient of diffusivity ($\nu = 0.01$) and f is the source term.

Applying the weighted residual method to reach the weak form and using integration by parts (having w=0 on the Dirichlet boundary) will lead to:

$$\int_0^1 (wau_x + w_x \nu u_x)dx = \int_0^1 wf dx$$

Introducing the discretization and using the Galerkin method, we obtain:

$$(C + K)u = F$$

where C is the convection matrix, K is the diffusion matrix, and F is the force matrix.

$$C = \int_0^1 N^T(a\frac{\partial N}{\partial x})dx, \quad K = \int_0^1 \nu\frac{\partial N^T}{\partial x}\frac{\partial N}{\partial x}dx, \quad F = \int_0^1 N^T f dx$$

Peclet number is introduced as the ration between convective and diffusive effects as: $Pe = \frac{ah}{2\nu}$, where h is the element size. The convection matrix C is unsymmetrix breaking the famous symmetry of the Galerkin method and leads to oscillatory solution if Pe is larger than 1 (convection domination).

To solve this issue, stabilization techniques are used in a consistent manner to make sure that the problem solved is the same. An extra term is added over the interior of all the elements; this term is a function of the differential equation residual to ensure consistency. This term is of the following form:

$$\sum_e \int_{\Omega_e} P(w)\tau R(u)d\Omega$$

where P is an operator applied to the test function, τ is a stabilization parameter, and R is the differential equation residual given as:

$$R(u) = au_x - \nu u_{xx} - f$$

Thus, the weak form using the stabilization techniques (SUPG and GLS) will be:

$$\int_0^1 (wau_x + w_x \nu u_x)dx + \sum_e \int_{\Omega_e} P(w)\tau R(u)d\Omega = \int_0^1 wf dx$$

For SUPG, the stabilization operator P is defined such that artificial diffusion is added only on the streamline direction in a consistent way.

$$P(w) = a.\nabla w$$

For GLS, the stabilization parameter P is defined by imposing least-squares formulation for the original differential equation.

$$P(w) = a.\nabla w - \nabla.(\nu \nabla w)$$

The GLS method is better than SUPG since it produces symmetric stabilization parameter which is easier to handle technically.

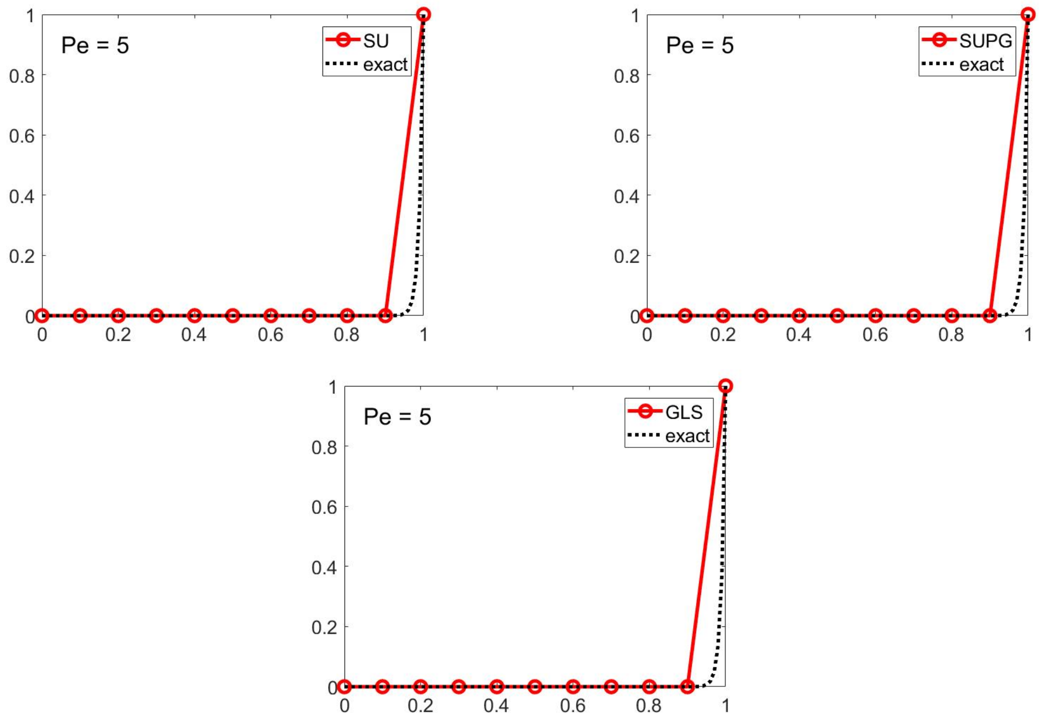
The stabilization parameter τ plays an important role in these techniques. Several proposals were done to calculate it. For 1d using linear elements, superconvergance is acheived with

$$\tau = \frac{h(coth(Pe) - 1/Pe)}{2a}$$

a different relation is obtained for quadratic elements. In addition, it should be noted that τ is a matrix but in simple cases can be just a scalar.

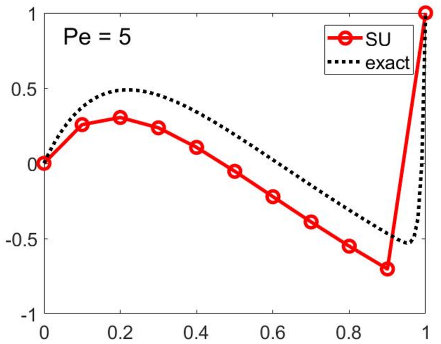
3 Different Source Term

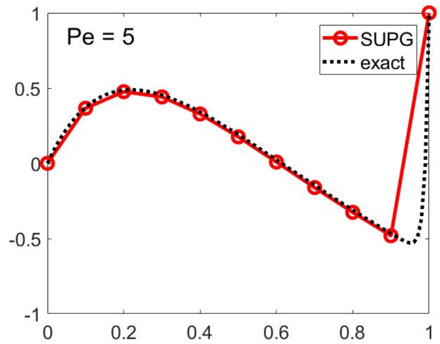
Using linear elements, two problems were solved: one having zero source term and another one having $f = 10e^{-5x} - 4e^{-x}$. The problems were solved using SU, SUPG and GLS methods. The basic results are shown below with 10 elements.

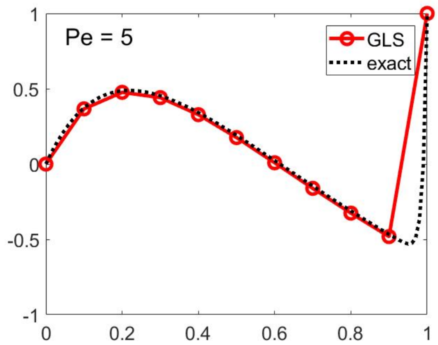


It should be noted that the three methods lead to the same exact result. The 3 methods are identical since the term that has 2^{nd} order derivative becomes zero due to using linear shape functions and the source term is zero which will lead to the same equation for all of them. These method with linear elements lead to exact nodal values as shown.

The results with the other source term is obtained below.





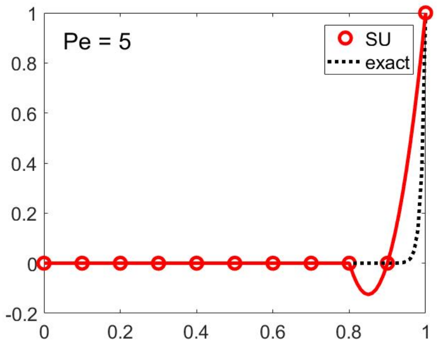


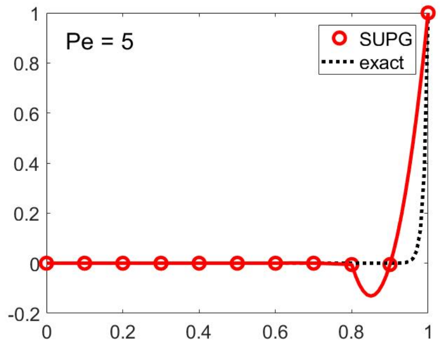
It should be noted from the results that GLS and SUPG are identical since the 2^{nd} order derivative is zero due to using linear shape functions. Morover, GLS and SUPG provide exact nodal solution. On the contrary, SU doesn't result in exact nodal values; its result was different due to the existence of a source term. SU doesn't take into account the residual in the formulation; that's why it's not a consistent method.

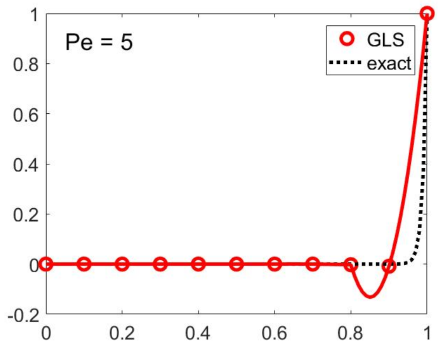
4 Using Quadratic Elements

The same problems were solved again but this time using quadratic elements. To do so, several parts of the code were modified to fit the change listed as:
1- The number of nodes and distance h were modified.
2- The connectivity matrix was modified.
3- Another value of τ was calculated and incorporated in the solution (outer nodes has different τ from the middle one).
4- Post Processing was modified. Instead of just connecting the nodes linearly, shape functions are used for the interpolation.

The results with zero source term are presented first.

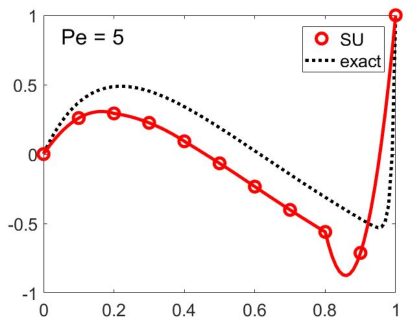


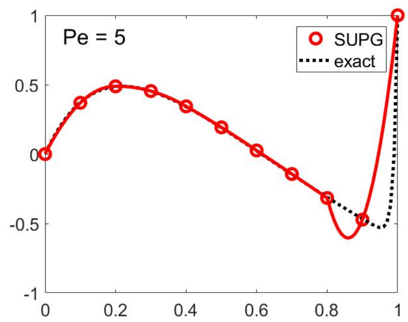


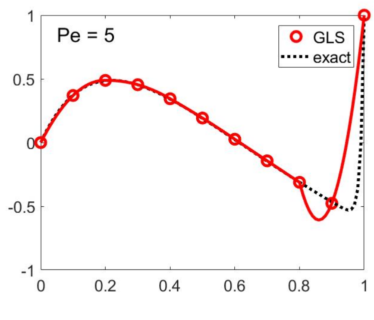


Clearly, the solution of all the methods is identical. Although, the formulation of the 3 methods is different, having no source term and the behaviour of the problem led to similar solution. Almost exact nodal values are obtained by all of the methods.

Solution with the source term is then obtained







Clearly, having a source term made things different as it affects the formulation. The SU fails to obtain exact nodal values. However, the SUPG and GLS methods have almost exact nodal values and have similar solution even though they have different stabilization terms.

5 Conclusion

Having convective term in the differential equation leads to instability issues. The classical Galerkin method fails to address the issue. Several ways are developed to solve this issue. Adding diffusion to the equation is the main solution. The Streamline-Upwind (SU) method applies the diffusion term in a non consistent way in the direction of streamline. The SUPG and GLS methods apply the diffusion in a consistent way by making the term function of the residual. Therefore, they have better approach to the problem.