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Perturbed Burger's Equation
$$\begin{cases} u_t^\varepsilon + u^\varepsilon u_x^\varepsilon = \varepsilon u_{xx}^\varepsilon & (x,t)\in[-1,1]\times[0,T] \\ u^\varepsilon(x,0) = u_0(x) & x\in[-1,1] \\ u^\varepsilon(-1,t) = u^\varepsilon(1,t) = 0 & t\in[0,T] \end{cases}$$

To simplify the notation, superscript "ε" is omitted in the following parts.

- One-step Taylor-Galerkin method.

$$u_t + f_x(u) = \varepsilon u_{xx} \quad \text{where } f(u)=\tfrac{1}{2}u^2 \quad (1)$$

Replace the first and second time derivatives in the Taylor expansion with spatial derivatives using equation (1). The gives.

$$u_t = -f_x + \varepsilon u_{xx}$$

$$u_{tt} = (-f_x + \varepsilon u_{xx})_t$$
$$= -(f_t)_x + \varepsilon (u_t)_{xx}$$
$$= -\left(\frac{\partial f}{\partial u}\frac{\partial u}{\partial t}\right)_x + \varepsilon(\varepsilon u_{xx} - f_x)_{xx}$$
$$= -[a(-f_x+\varepsilon u_{xx})]_x + \varepsilon^2 u_{xxxx} - \varepsilon f_{xxx}$$
$$= (af_x - \varepsilon a u_{xx})_x + \varepsilon^2 u_{xxxx} - \varepsilon f_{xxx} \quad (2)$$

Introduce (2) into the Taylor series expansion.

$$\frac{u^{n+1}-u^n}{\Delta t} = -f_x^n + \varepsilon u_{xx}^n + \frac{\Delta t}{2}\left[(af_x - \varepsilon a u_{xx}^n)_x + \varepsilon^2 u_{xxxx}^n - \varepsilon f_{xxx}^n\right]$$

Apply the Galerkin formulation in space, we have.

$$\left(w, \frac{u^{n+1}-u^n}{\Delta t}\right) = (w_x, f^n) - \langle wn, f^n\rangle - (w_x, \varepsilon u_x^n) + \langle wn, \varepsilon u_x^n\rangle$$
$$- \frac{\Delta t}{2}(w_x, af_x^n - \varepsilon a u_{xx}^n) + \frac{\Delta t}{2}\langle wn, af_x^n - \varepsilon a u_{xx}^n\rangle$$
$$+ \frac{\Delta t}{2}\varepsilon^2 \underbrace{(w, u_{xxxx})}_{①} - \frac{\Delta t}{2}\varepsilon \underbrace{(w, f_{xxx})}_{②} \quad (3)$$

where $n = 1/-1$ for 1D problem. $(\cdot,\cdot)$ integration over domain, $\langle\cdot,\cdot\rangle$ integration along the boundary.

$$① = -(w_x, u_{xxx}^n) + \langle wn, u_{xxx}^n\rangle$$
$$= (w_{xx}, u_{xx}^n) - \langle w_x n, u_{xx}^n\rangle + \langle wn, u_{xxx}^n\rangle$$
$$= \text{B.C. terms} + (w_{xx}, u_{xx}^n) \quad (4)$$

$$② = -(w_x, f_{xx}^n) + \langle wn, f_{xx}^n\rangle \quad (5)$$

Subsititute (4), (5) into (3) we will have the final form of the Galerkin formulation. The one-step method is not trivial in implementation as we need to carefully choose the shape functions in order to make sure (w_{xx}, u_{xx}) in (4) and (w_x, f_{xx}) are integrable and have physical meaning

If we turn to Two-step Taylor-Galerkin, we will observe that only first order derivative appears. Therefore it is easier to apply two-step method in this case.

- Two-step Taylor-Galerkin method

$$u^{n+1/2} = u^n + \frac{\Delta t}{2} u_t^n = u^n - \frac{\Delta t}{2} f_x^n + \frac{\Delta t}{2} \varepsilon u_{xx}^n$$

$$u^{n+1} = u^n + \Delta t\, u_t^{n+1/2} = u^n - \Delta t\, f_x^{n+1/2} + \Delta t\, \varepsilon\, u_{xx}^{n+1/2}$$

Where $f^n = f(u^n)$, $f^{n+1/2} = f(u^{n+1/2})$

In weak form, the problem in the second integration step is

$$\left(w, \frac{u^{n+1} - u^n}{\Delta t}\right) = (w_x, f^{n+1/2}) - \langle wn, f^{n+1/2} \rangle + \Delta t\, \varepsilon \langle wn, u_x^{n+1/2} \rangle$$
$$- \Delta t\, \varepsilon\, (w_x\, u_x^{n+1/2})$$