Class Homework 4: 2D Unsteady Convection-Diffusion

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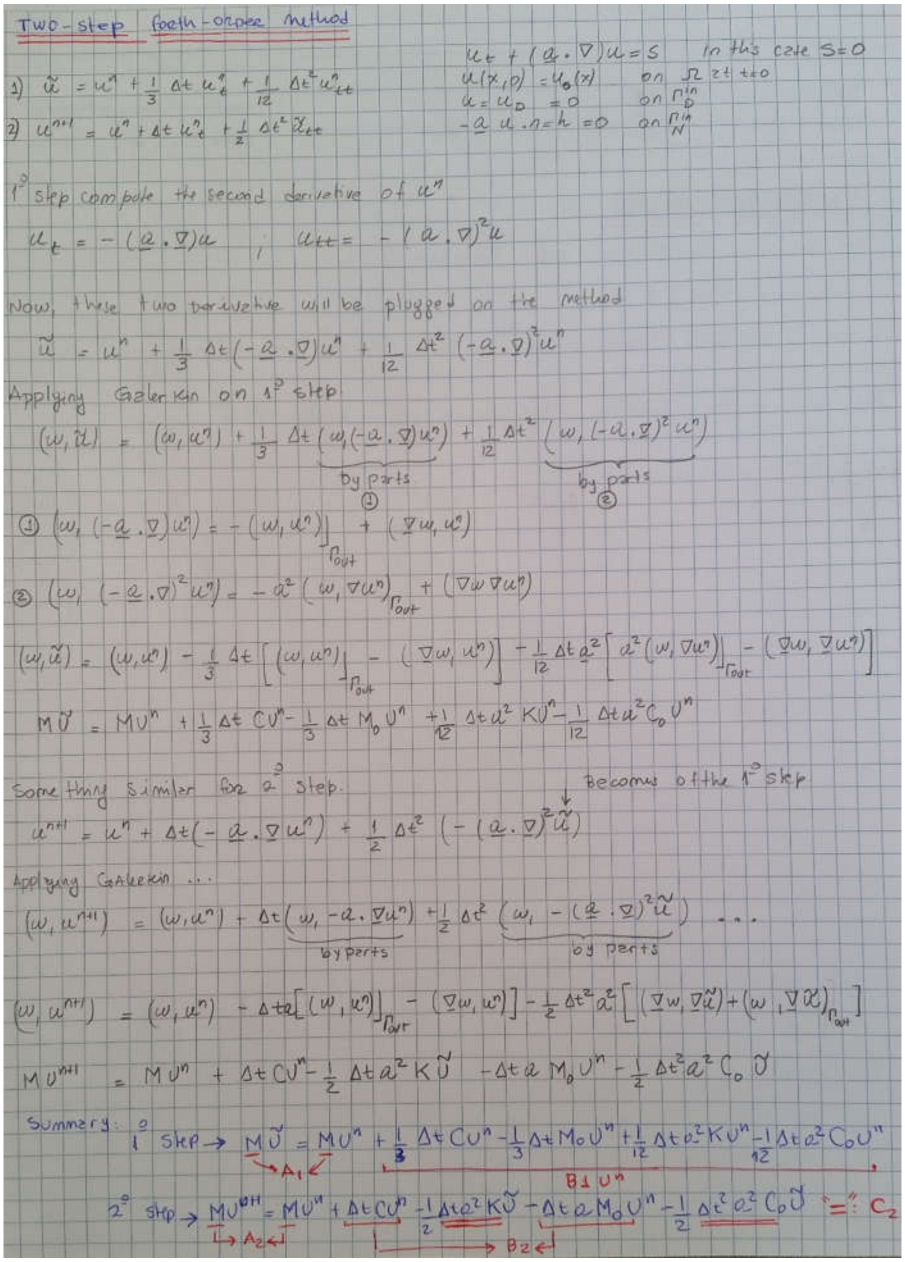
Two-step fourth-order method:

Before implementing this method was necessary developed by hand. The Figure 1, shows step by step how the method was obtained by hand.



Two-step foeth-order method

$u_t + (\underline{a}\cdot\nabla)u = S$ in this case $S=0$

$u(x,0) = u_0(x)$ on Ω at $t=0$

$u = u_D = 0$ on Γ_D^{in}

$-\underline{a}\, u \cdot n = h = 0$ on Γ_N^{in}

1) $\tilde{u} = u^n + \frac{1}{3}\Delta t\, u_t^n + \frac{1}{12}\Delta t^2 u_{tt}^n$

2) $u^{n+1} = u^n + \Delta t\, u_t^n + \frac{1}{2}\Delta t^2 \tilde{u}_{tt}$

1° step compute the second derivative of u^n

$u_t = -(\underline{a}\cdot\nabla)u \quad , \quad u_{tt} = +(\underline{a}\cdot\nabla)^2 u$

Now, these two derivative will be plugged on the method

$$\tilde{u} = u^n + \frac{1}{3}\Delta t(-\underline{a}\cdot\nabla)u^n + \frac{1}{12}\Delta t^2(-\underline{a}\cdot\nabla)^2 u^n$$

Applying Galerkin on 1° step

$$(w,\tilde{u}) = (w,u^n) + \frac{1}{3}\Delta t \underbrace{(w,(-\underline{a}\cdot\nabla)u^n)}_{\text{by parts } ①} + \frac{1}{12}\Delta t^2 \underbrace{(w,(-\underline{a}\cdot\nabla)^2 u^n)}_{\text{by parts } ②}$$

① $(w,(-\underline{a}\cdot\nabla)u^n) = -(w,u^n)\big|_{\Gamma_{out}} + (\nabla w, u^n)$

② $(w,(-\underline{a}\cdot\nabla)^2 u^n) = -a^2(w,\nabla u^n)_{\Gamma_{out}} + (\nabla w\, \nabla u^n)$

$$(w,\tilde{u}) = (w,u^n) - \frac{1}{3}\Delta t\left[(w,u^n)\big|_{\Gamma_{out}} - (\nabla w, u^n)\right] - \frac{1}{12}\Delta t\, \underline{a}^2\left[a^2(w,\nabla u^n)\big|_{\Gamma_{out}} - (\nabla w,\nabla u^n)\right]$$

$$M\tilde{U} = MU^n + \frac{1}{3}\Delta t\, CU^n - \frac{1}{3}\Delta t\, M_0 U^n + \frac{1}{12}\Delta t\, a^2 KU^n - \frac{1}{12}\Delta t\, a^2 C_0 U^n$$

Some thing similar for 2° step. Becomes of the 1° step

$$u^{n+1} = u^n + \Delta t(-\underline{a}\cdot\nabla u^n) + \frac{1}{2}\Delta t^2\left(-(\underline{a}\cdot\nabla)^2\tilde{u}\right)$$

Applying Galerkin ...

$$(w,u^{n+1}) = (w,u^n) + \Delta t\underbrace{(w,-\underline{a}\cdot\nabla u^n)}_{\text{by parts}} + \frac{1}{2}\Delta t^2\underbrace{(w,-(\underline{a}\cdot\nabla)^2\tilde{u})}_{\text{by parts}} \ldots$$

$$(w,u^{n+1}) = (w,u^n) - \Delta t\, a\left[(w,u^n)\big|_{\Gamma_{out}} - (\nabla w, u^n)\right] - \frac{1}{2}\Delta t^2 a^2\left[(\nabla w,\nabla\tilde{u}) + (w,\nabla\tilde{u})_{\Gamma_{out}}\right]$$

$$MU^{n+1} = MU^n + \Delta t\, CU^n - \frac{1}{2}\Delta t\, a^2 K\tilde{U} - \Delta t\, a\, M_0 U^n - \frac{1}{2}\Delta t^2 a^2 C_0 \tilde{U}$$

Summary:

1° step $\rightarrow$ $\underbrace{M}_{A_1}\tilde{U} = \underbrace{MU^n + \frac{1}{3}\Delta t\, CU^n - \frac{1}{3}\Delta t\, M_0U^n + \frac{1}{12}\Delta t\, a^2 KU^n - \frac{1}{12}\Delta t\, a^2 C_0 U^n}_{B_1 U^n}$

2° step $\rightarrow$ $\underbrace{M}_{A_2}U^{n+1} = MU^n + \Delta t\, CU^n - \frac{1}{2}\Delta t\, a^2 K\tilde{U} - \Delta t\, a\, M_0 U^n - \frac{1}{2}\Delta t^2 a^2 C_0\tilde{U} =: C_2$ (B_2)

Figure 1: Method developed by hand.

The next figure shows the line of the code where this method was implemented.

```
elseif meth == 8 % Two-Step fourth-order method
    % First Step
    A1 = M;
    B1 = (1/3)*C*dt - (1/12)*K*dt^2 - (1/3)*Mo*dt + (1/12)*Co*dt^2;
    f1 = (1/3)*v1*dt + (1/12)*v2*dt^2 - (1/12)*vo*dt^2;
    %Second Step
    A2 = M;
    B2 = C*dt - Mo*dt;
    C2 = -(1/2)*K*dt^2 + (1/2)*Co*dt^2;
    f2 = v1*dt + (1/2)*v2*dt^2 - (1/2)*vo*dt^2;
```

Figure 2: Two-step fourth-order method implemented.

In the code can be observed all the matrices that was developed by hand.

<u>Result:</u>

The problem parameters used in order to show results obtained by using different method were:

<u>Parameters of the problem:</u>
v(x,y) = (-y,x) (convective velocity)
h = 0.04 (mesh size)
25 bilinear quadrilateral elements (Quad4) per side were used.
S = 0 (source term)

<u>Initial and boundary condition:</u>
Cosine hill was used as initial condition.
Homogeneous Dirichlet condition on the inlet boundary was used.

Eight methods were used in order to computed the solution and see which solution the best is.

<u>Galerkin formulation + Lax-Wendroff method with with consistent mass matrix (TG2)</u>

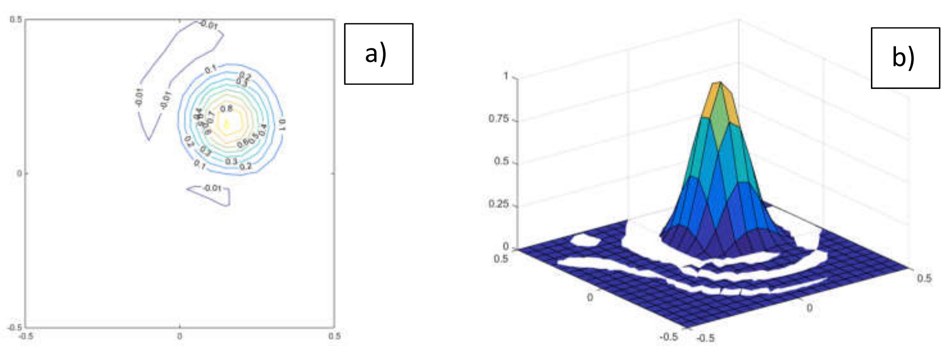


Figure 3: Solution obtained by TG2 a) Isolines, b) 3D plot

Galerkin formulation + Lax-Wendroff method with lumped mass matrix (LW-FD)

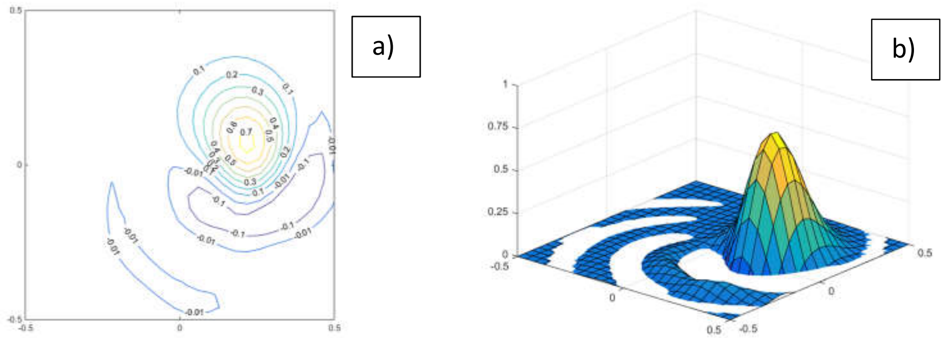


Figure 4: Solution obtained by LW-FD a) Isolines, b) 3D plot

Galerkin formulation + third-order explicit Taylor-Galerkin method (TG3)

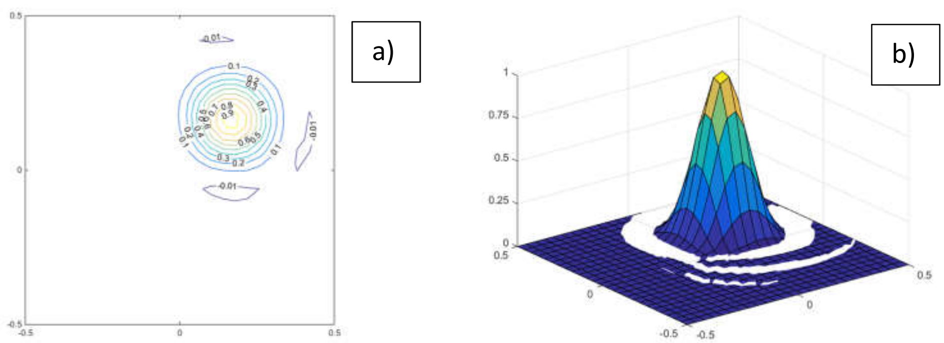


Figure 5: Solution obtained by TG3 a) Isolines, b) 3D plot

Galerkin formulation + Crank-Nicolson with consistent mass matrix (CN)

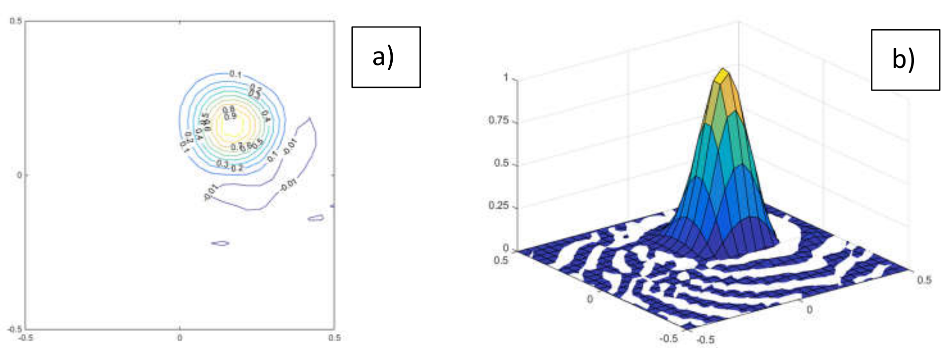


Figure 6: Solution obtained by CN a) Isolines, b) 3D plot

Galerkin formulation + Crank-Nicolson with lumped mass matrix (CN-FD)

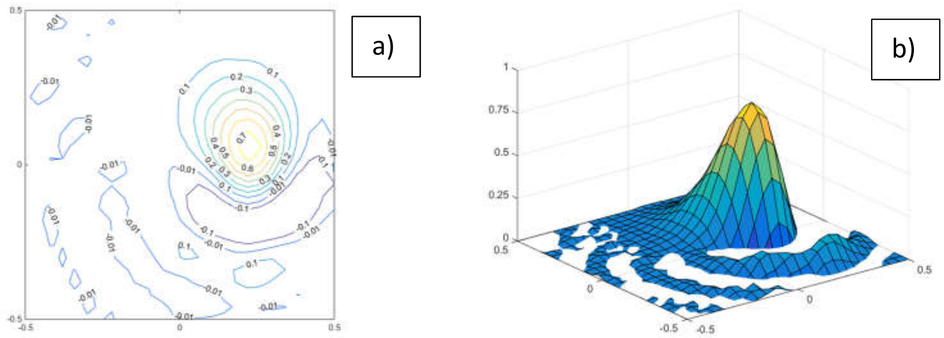


Figure 7: Solution obtained by CN-FD a) Isolines, b) 3D plot

Least-squares formulation + Crank-Nicolson method (Carey-Jiang)

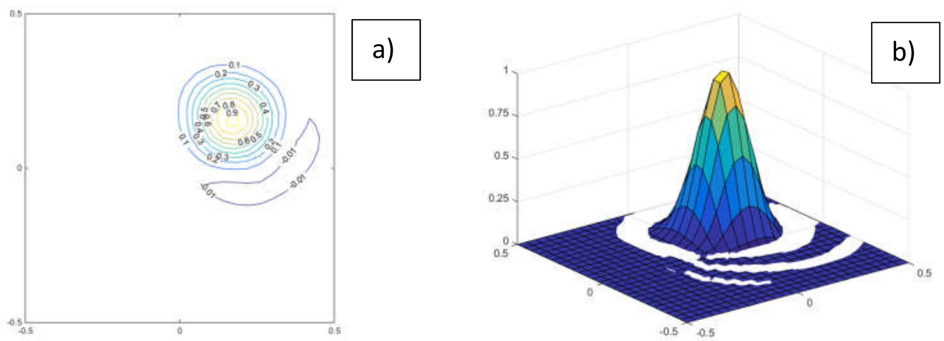


Figure 8: Solution obtained by CJ a) Isolines, b) 3D plot

Galerkin formulation + third order two-step Taylor-Galerkin method (TG3-2S)

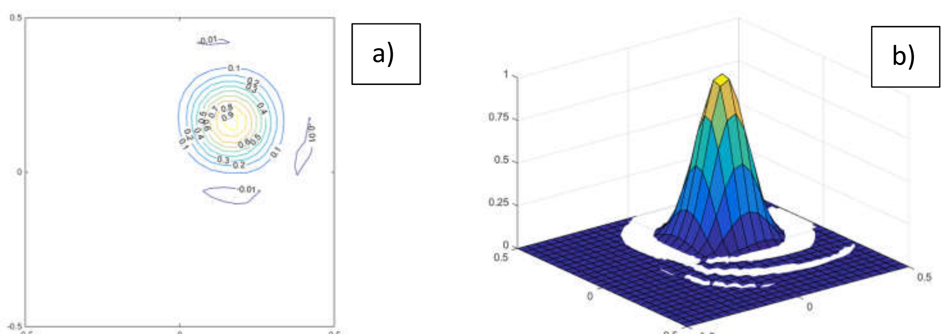


Figure 9: Solution obtained by TG3-2S a) Isolines, b) 3D plot

Two-step fourth-order (TG4-2S)

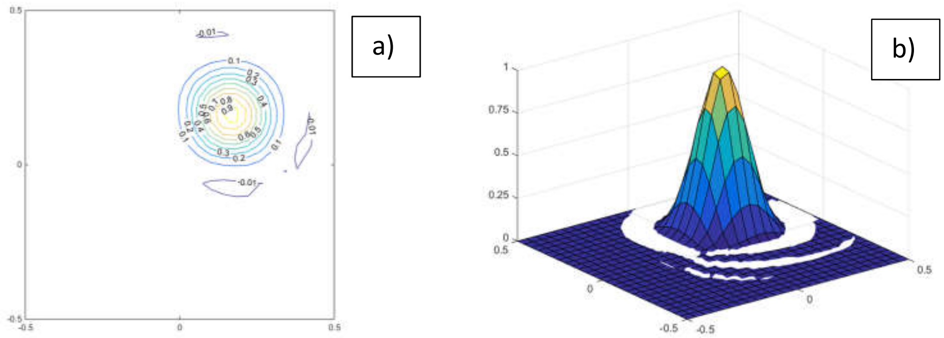


Figure 10: Solution obtained by TG4-2S a) Isolines, b) 3D plot

To compare the accuracy of the various methods, explicit and implicit, the maximum and minimum values of the computed solution are provided in Table 1.

Table 1: Max and min results obtain using different methods.

Method	Numerical solution		Exact solution		Error %	
	u_{max}	u_{min}	u_{max}	u_{min}	$\Delta error\_u_{max}$	$\Delta error\_u_{min}$
TG2	0.917	-0.0305	1	0	8.3	3.05
LW-FD	0.7340	-0.1947	1	0	27	19.5
TG3	0.9351	-0.0225	1	0	6.5	2.3
CN	0.9855	-0.0541	1	0	1.5	5.4
CN-FD	0.7618	-0.0249	1	0	23.8	2.5
CJ	0.9307	-0.0305	1	0	6.3	3.1
TG3-2S	0.9375	-0.0216	1	0	6.3	2.2
TG4-2S	0.9366	-0.0216	1	0	6.3	2.2

It can be seen that the worst result obtained was that correspond by using LW-FD.

According to Donae and Huerta's book, it can be split these method into explicit and implicit. The explicit methods used were LW-FD, TG2, TG3, TG3-2S and TG4-2S. The greater accuracy of the finite element schemes employing TG2, TG3, TG3-2S and TG4-2S is clearly apparent, being TG3-2S and TG4+2S the bests methods. On the other hand, the implicit methods used were CN, CJ and CN-FD. CN and CJ gave the best results. It can be observed that the Crank-Nicholson scheme with the Galerking formulation represent a non-dissipative method in pure convection. As a result, significant non-physical oscillation can be seen in Figure 6 and Figure 7. These oscillations were attenuated using a dissipative spatial formulation, such as the least-square FEM of Carey and Jiang. The result can be noted in Figure 8.