Part 1: Pure transport equation

$$u_t + (\underline{a}\cdot\nabla)u = 0$$

In the examples

- Zero source term
- Dirichlet boundary conditions on the inflow boundary

(1) θ-method (Crank-Nicolson $\theta = 1/2$)

Time discretization of C-N method is

$$\frac{\Delta u}{\Delta t} + \frac{1}{2}(\underline{a}\cdot\nabla)\Delta u = \frac{1}{2}\cancel{s^{n+1}}^{0} + \frac{1}{2}\cancel{s^{n}}^{0} - \underline{a}\cdot\nabla u^n \quad ① \text{ no source}$$

Galerkin formulation

Integrate ① with weight function we have

$$(w, \frac{\Delta u}{\Delta t}) + \frac{1}{2}(w, (\underline{a}\cdot\nabla)\Delta u) = -\langle w, \underline{a}\cdot\nabla u^n\rangle$$

Integration by parts

$$(w, \frac{\Delta u}{\Delta t}) - \frac{1}{2}(\nabla w, \underline{a}\Delta u) = (\nabla w, \underline{a}u^n) - \langle(\underline{a}\cdot\underline{n})w, u^n\rangle_{\Gamma}$$

Boundary condition. In the example we studied, $u_{out} = 0$. actually. So

$$(w, \frac{\Delta u}{\Delta t}) - \frac{1}{2}(\nabla w, \underline{a}\Delta u) + \frac{1}{2}\cancel{\langle(\underline{a}\cdot\underline{n})w, \Delta u\rangle_{\Gamma out}}^{0} = (\nabla w, \underline{a}u^n) - \cancel{\langle(\underline{a}\cdot\underline{n})w, u^n\rangle_{\Gamma out}}^{0}$$

FE discretization

$$\left(\frac{1}{\Delta t}\underset{\sim}{M} + \frac{1}{2}\underset{\sim}{C}\right)\Delta\underline{u} = \cancel{\underline{f}}^{0} - \underset{\sim}{C}\underline{u}^n$$

where

$\underset{\sim}{M}$ $\quad \int_\Omega N_a N_b \, d\Omega$

$\underset{\sim}{C}$ $\quad \int_\Omega N_a(\underline{a}\cdot\nabla N_b)\, d\Omega$.

$\underset{\sim}{B}_{out}$ is zero for this case

(2) Lax-Wendroff (TG2)

Time discretization of L-W method is

$$\frac{\Delta u}{\Delta t} = -\underline{a}\cdot\nabla u^n + \frac{\Delta t}{2}(\underline{a}\cdot\nabla)^2 u^n \quad ②$$

Galerkin formulation

Integrate ② with weight function, we have

$$(w, \frac{\Delta u}{\Delta t}) = -(w, \underline{a}\cdot\nabla u^n) + \frac{\Delta t}{2}(w, (\underline{a}\cdot\nabla)^2 u^n)$$

Integration by parts

$$(w, \frac{\Delta u}{\Delta t}) = (\underline{a}\cdot\nabla w, u^n) - \frac{\Delta t}{2}(\underline{a}\cdot\nabla w, \underline{a}\cdot\nabla u^n) + \langle(\underline{a}\cdot\underline{n})w, u^n\rangle - \frac{\Delta t}{2}\langle(\underline{a}\cdot\underline{n})w, (\underline{a}\cdot\nabla)u^n\rangle$$

FE discretization similiar to C-N method

$$\frac{1}{\Delta t}\underset{\sim}{M}\Delta\underline{u} = \left(-\Delta t\,\underset{\sim}{C} - \frac{\Delta t}{2}\underset{\sim}{K} + \cancel{\underset{\sim}{B}_{out}}^{0}\right)\underline{u}^n + \cancel{\underline{f}}^{0} \qquad \text{no souce, } u^n|_{\Gamma out} = 0.$$

where $\underset{\sim}{M}$ $\underset{\sim}{C}$ is the same as in C-N.

$\underset{\sim}{K}$ $\quad \int_\Omega (\underline{a}\cdot\nabla N_a)(\underline{a}\cdot\nabla N_b)\, d\Omega$

(3) TG3.

Time distetization of TG3 method is

$$\left[1-\frac{\Delta t^2}{6}(\underset{\sim}{a}\cdot\nabla)^2\right]\frac{\Delta u}{\Delta t} = -\underset{\sim}{a}\cdot\nabla u^n + \frac{\Delta t}{2}(\underset{\sim}{a}\cdot\nabla)^2 u^n \quad ③$$

Galerkin formulation

- Integrate ③ with weight function, we have

$$(w, \frac{\Delta u}{\Delta t}) - (w, \frac{\Delta t^2}{6}(\underset{\sim}{a}\cdot\nabla)^2\frac{\Delta u}{\Delta t}) = -(w, a\nabla u^n + \frac{\Delta t}{2}(w, \frac{\Delta t}{2}(\underset{\sim}{a}\cdot\nabla)^2 u^n)$$

Integration by parts, neglect the boundary terms

$$(w, \frac{\Delta u}{\Delta t}) + (a\nabla w, \frac{\Delta t}{6}(\underset{\sim}{a}\cdot\nabla)\Delta u) = -(w, a\nabla u^n) - \frac{\Delta t}{2}(\underset{\sim}{a}\cdot\nabla w, \underset{\sim}{a}\cdot\nabla u^n)$$

no source, $\underset{\sim}{B}_{out} = 0$

FE discretization

$$(\underset{\sim}{M} + \frac{\Delta t^2}{6}\underset{\sim}{K})\Delta \underline{U} = (-\Delta t\,\underset{\sim}{C} - \frac{\Delta t^2}{2}\underset{\sim}{K})\underline{U}^n$$

where $\underset{\sim}{M}$, $\underset{\sim}{C}$, $\underset{\sim}{K}$ are the same as discribed before.

Part 2 Non-linear System (Burger's Equation)

$$\begin{cases} u_t + u u_x = \varepsilon u_{xx} \\ u(x,0) = u_0(x) \end{cases}$$

Weak form:

$$\int_0^L w u_t\,dx + \int_0^L w u u_x\,dx = \varepsilon\int_0^L w u_{xx}\,dx$$

Integration by parts and neglect the boundary terms

$$\int_0^L w u_t\,dx + \int_0^L w u u_x\,dx + \varepsilon\int_0^L w_x u_x\,dx = 0.$$

FE discretization.

$$\underset{\sim}{M}\dot{\underline{u}} + \underset{\sim}{C}(\underline{u})\underline{u} + \varepsilon\underset{\sim}{K}\underline{u} = \underline{0}$$

Where $\underset{\sim}{M}$. $\underset{\sim}{C}$, $\underset{\sim}{K}$ are the same as discribed before.

For Backward-Euler, we have

$$\underset{\sim}{M}\frac{\underline{u}^{n+1}-\underline{u}^n}{\Delta t} + \underset{\sim}{C}(\underline{u}^{n+1})\underline{u}^{n+1} + \varepsilon\underset{\sim}{K}\underline{u}^{n+1} = \underline{0}$$

$$\Rightarrow (\underset{\sim}{M} + \Delta t(\underset{\sim}{C}(\underline{u}^{n+1}) + \varepsilon\underset{\sim}{K}))\underline{u}^{n+1} = \underset{\sim}{M}\underline{u}^n \text{ Non-linear system.}$$

Newton-Raphson Method

$$\underline{u}^{n+1}_{k+1} = \underline{u}^{n+1}_k - \underset{\sim}{J}^{-1}(\underline{u}^{n+1}_k) f(\underline{u}^{n+1}_k)$$

Where

$$f(\underline{u}) = (\underset{\sim}{M} + \Delta t(\underset{\sim}{C}(\underline{u}^{n+1}) + \varepsilon\underset{\sim}{K}))\underline{u}^{n+1} - \underset{\sim}{M}\underline{u}^n$$

$$\underset{\sim}{J} = \frac{df}{d\underline{u}} = \underset{\sim}{M} + \Delta t(\underset{\sim}{C}(\underline{u}^{n+1}) + \varepsilon\underset{\sim}{K}) + \Delta t\,\frac{\partial\underset{\sim}{C}(\underline{u}^{n+1})}{\partial\underline{u}}\cdot\underline{u}^{n+1}$$

$$\underset{\sim}{C} \leftarrow \int w u u_x\,dx$$

$$D(u u_x)[\delta u] = \underbrace{\delta u\, u_x}_{\frac{\partial \underset{\sim}{C}}{\partial \underline{u}}\cdot\underline{u}} + \underbrace{u\,\delta u_x}_{\underset{\sim}{C}}$$

Therefore: $\left[\frac{\partial \underset{\sim}{C}}{\partial \underline{u}}\cdot\underline{u}\right]^e_{ab} = \int_{\Omega_e}\underline{N}^T(u_x)\underline{N}\,d\Omega = \int_{\Omega_e}\underline{N}^T(\underline{N}_x \underline{u}_e)\underline{N}\,d\Omega.$