Finite Elements in Fluid

Homework 4a: Unsteady-transport examples

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1. INTRODUCTION

1.1 Pure transport equation

$$\begin{aligned} &u_t + (\boldsymbol{a} \bullet \nabla u) = s && in\ \Omega \times [0,T] \\ &u(\boldsymbol{x},0) = u_0(\boldsymbol{x}) && on\ \Omega\ at\ t = 0 \\ &u = u_D && on\ \Gamma_D^{in} \times [0,T] \\ &-\boldsymbol{a}u \bullet \boldsymbol{n} = h && on\ \Gamma_D^{in} \times [0,T] \end{aligned}$$

$$\Gamma^{in} = \{x \in \Gamma | \boldsymbol{a} \bullet \boldsymbol{n} < \boldsymbol{0}\}$$

In the examples:

Zero source term

Dirichlet boundary conditions on the inflow boundary

1.2 method

1.2.1 θ-method

$$\frac{\triangle u}{\triangle t} + \theta(\boldsymbol{a} \bullet \nabla u) \triangle u = \theta s^{n+1} + (1-\theta)s^n - a \bullet \nabla u^n$$

Crank-Nicolson: $\theta = 1/2$

1.2.2 Lax-Wendroff

$$\frac{\triangle u}{\triangle t} = -\boldsymbol{a} \bullet \nabla u^n + \frac{\triangle t}{2}(\boldsymbol{a} \bullet \nabla)^2 u^n + s^n + \frac{\triangle t}{2}(s_t^n - \boldsymbol{a} \bullet \nabla s^n)$$

1.2.3 Third order Taylor-Galerkin

$$\begin{aligned} &\left[1 - \frac{\triangle t^2}{6}(\boldsymbol{a} \bullet \nabla)^2\right]\frac{\triangle u}{\triangle t} \\ &\qquad = -\boldsymbol{a} \bullet \nabla u^n + \frac{\triangle t}{2}(\boldsymbol{a} \bullet \nabla)^2 u^n + s^n + \frac{\triangle t}{2}(s_t^n - \boldsymbol{a} \bullet \nabla s^n) \\ &\qquad + \frac{\triangle t^2}{6}(s_{tt}^n - \boldsymbol{a} \bullet \nabla s_t^n) \end{aligned}$$

2. OBJECTIVE

2.1 Solve the following exercise.

$$\begin{cases} u_t + a u_x = 0 & x \in (0,1), t \in (0,0.6] \\ u(\boldsymbol{x},0) = u_0(\boldsymbol{x}) & x \in (0,1) \\ u(0,t) = 1 & x \in (0,0.6] \end{cases}$$

$$u_0(\boldsymbol{x}) = \begin{cases} 1 & if\ x \leq 0.2, \\ 0 & otherwise \end{cases}$$

$$a = 1, \triangle x = 2 \cdot 10^{-2}, \triangle t = 1.5 \cdot 10^{-2}$$

2.2 Compute the courant number.

2.3 Solve the problem using the Crank-Nicholson scheme in time and linear finite element for the Galerkin scheme in space. Discussing the solution accurate.

2.4 Solve the problem using the second-order Lax-Wendroff method. Discussing whether the solution is accurate or not. Change it and comment the results.

2.5 Solve the problem using the third-order explicit Taylor Galerkin method. Comment the result.

3. METHODOLOGY AND RESULTS

3.1 The courant number $C = |a|\Delta t/h$, where a is convection coefficient, Δt is length of time steps and h is the length of space. For the case, the number of time steps is 40, due to $0.6/(1.5 \cdot 10^{-2})$. And courant number can be calculated as 0.75.

3.2 From the result, we can find the neither the consistent matrix or the lumped one can get the accurate solution. The oscillation due to the Galerkin formulation and Crank Nicolson is not such a monotone scheme. Using nonlinear viscosity added in front improve the scheme locally in first order accurate.

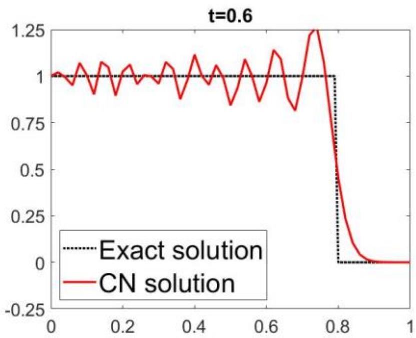

Figure 1. Crank Nicolson Consistent Matrix at t=0.6s

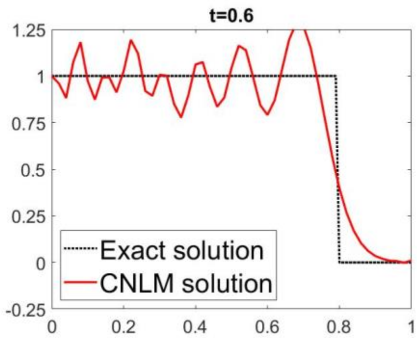

Figure 2. Crank Nicolson Lumped Matrix at t=0.6s

3.3 Since the Courant number is large than the stability range of Lax-Wendroff, we can not expect the solution to be accurate. To get a better solution, we can lumpe mass matrix on Lax-Wendroff. This method can increase the stability range of Lax-Wendroff(TG2).

Table 3.6 Stability limits for different schemes

Scheme	Stability limits	
	1D	2D
CN	unconditional stability	
TG2	$C^2 \leq 1/3$	$c_x^{2/3} + c_y^{2/3} \leq (1/3)^{1/3}$
TG3	$C^2 \leq 1$	$c_x^{2/3} + c_y^{2/3} \leq 1$
TG3-2S	$C^2 \leq 3/4$	$c_x^2 + c_y^2 \leq 3/4$
CJ	unconditional stability for $1/2 \leq \theta \leq 1$	

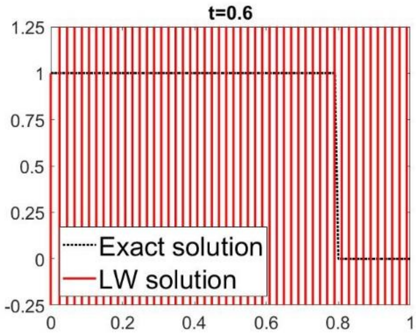

Figure 3. Lax-Wendroff Consistent Matrix at t=0.6s

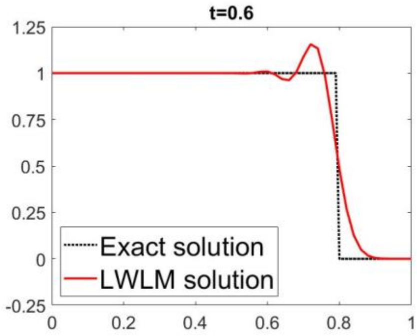

Figure 4. Lax-Wendroff Lumped Matrix at t=0.6s

To get the lumped mass matrix, we should prepare a diagonal matrix with the diagonal terms which are the sum of its rows' component.

```
function ML=MatrixLumped(M)

sizeM=size(M);
ML=zeros(sizeM(1),sizeM(2));
for i=1:sizeM(1)
    LumpedComponent=0;
    for j=1:sizeM(2)
        LumpedComponent=LumpedComponent+M(i,j);
    end
    ML(i,i)=LumpedComponent;
end
```

Add codes in main function to implement the Matrixlumped function.

```
 if method==2 || method==4
    ML=MatrixLumped(M);
   M=ML;
end
```

3.4 The third-order explicit Taylor-Galerkin method shows smoothly and gets better accuracy than Lax-Wendroff with lumped matrix. Due to the Galerkin formulation, the steep functions cannot be shaped. Compared with TG2, this method blur and reduce the oscillation range.

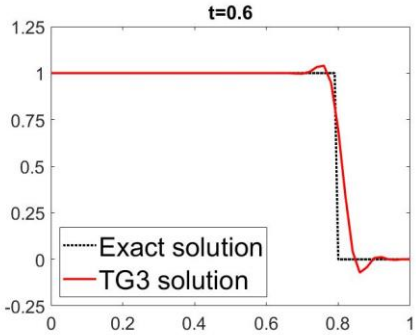

Figure 5. TG3 Consistent Matrix at t=0.6s

4. Conclusion

Crank-Nicholson method cannot keep stability in this case whether it is consistent matrix or lumped matrix. For Lax Lax-Wendroff(TG2), the stability range is $C \leq \sqrt{1/3}$. We can add the lumped mass matrix and obtain the stability solution. It is meant that stability range increase when using a lumped mass matrix representation. It is obviously that when keeping C in the stability range, results are more accurate if the consistent mass matrix representation is used.

5. Reference

[1] Lecture slides in Finite elements in fluid.

[2] Finite Element Methods for Flow Problems, Jean Donea and Antonio Huerta.