<u>Class Homework 5: 1D Unsteady pure convection</u>

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The equation to be solved correspond to the 1D unsteady pure convection. The source term is s=0 and there is no Neumann bc. The Initial condition is a steep front.

$$\begin{cases} u_t + a u_x = 0 & x \in (0,1),\ t \in (0, 0.6] \\ u(x,0) = u_0(x) & x \in (0,1) \\ u(0,t) = 1 & t \in (0, 0.6] \end{cases}$$

$$u_0(x) = \begin{cases} 1 & \text{if } x \le 0.2, \\ 0 & \text{otherwise} \end{cases}$$

$$a = 1, \Delta x = 2 \cdot 10^{-2}, \Delta t = 1.5 \cdot 10^{-2}$$

A mesh of uniform linear elements of size h = 1/50 will be used. The two Figure below shows the implementation Crank-Nicholson and least-square formulation. One by hand, the other by code.

$$\left(\frac{w}{\Delta t} + \frac{1}{2}\underline{a}\cdot\nabla w\ ;\ \frac{\Delta u}{\Delta t} + \frac{1}{2}\underline{a}\cdot\nabla\Delta u\right)$$

$$\left(\frac{w}{\Delta t}, \frac{\Delta u}{\Delta t}\right) = \frac{1}{\Delta t^2}(w, \Delta u) = \frac{M}{\Delta t^2}\Delta u$$

$$\left(\frac{1}{2}\underline{a}\cdot\nabla w, \frac{\Delta u}{\Delta t}\right) = \frac{1}{2\Delta t}\int_{\Omega}\underline{a}\cdot\nabla w\,\Delta u\,d\Omega = \frac{-1}{2\Delta t}C$$

$$\left(\frac{w}{\Delta t}, \frac{1}{2}\underline{a}\cdot\nabla u\right) = \frac{1}{2\Delta t}C$$

$$\left(\frac{1}{2}\underline{a}\cdot\nabla w, \frac{1}{2}\underline{a}\cdot\nabla u\right) = \frac{1}{4}a^2 K$$

$$\left(\frac{w}{\Delta t} + \frac{1}{2}\underline{a}\cdot\nabla w, -\underline{a}\cdot\nabla u^n\right)$$

$$\left(\frac{w}{\Delta t}, -\underline{a}\cdot\nabla u^n\right) = -\frac{1}{\Delta t}\underline{a}\,C u^n$$

$$\frac{1}{2}\underline{a}\cdot\nabla w, -\underline{a}\,\nabla u^n = -\frac{1}{2}a^2 K u^n$$

$$\frac{M}{\Delta t^2}\Delta u - \frac{1}{2\Delta t}C\Delta u + \frac{1}{2\Delta t}C\Delta u + \frac{1}{4}a^2 K\Delta u = \frac{-1}{\Delta t}\underline{a}\,C u^n - \frac{1}{2}a^2 K u^n$$

$$M\Delta u - \frac{\Delta t}{2}C\Delta u + \frac{\Delta t}{2}C\Delta u + \frac{1}{4}\Delta t^2 a^2 K\Delta u = -\Delta t\,\underline{a}\,C u^n - \frac{\Delta t^2}{2}a^2 K u^n \quad (2)$$

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Nx = Nxi_mef(ig,:)*2/h;
w_ig = weight(ig);
x = xm + h/2*xipg(ig); % x-coordinate of the current Gauss point
% Matrices assembly

A(isp,isp) = A(isp,isp) + w_ig*(N'*N + dt_2*((N'*(a*Nx))+(N'*(a*Nx))')+...
                                (dt^2/4)*(a*Nx)'*(a*Nx));
B(isp,isp) = B(isp,isp) - w_ig*(dt*(N'*(a*Nx))+(dt^2/2)*(a*Nx)'*(a*Nx));
f(isp) = f(isp) + w_ig*(N')*SourceTerm(x);
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Figure 1 shows the results for Crank-Nicholson scheme in time and Galerkin formulation and Crank-Nicholson scheme in time and the least-squares formulation in space.

Results

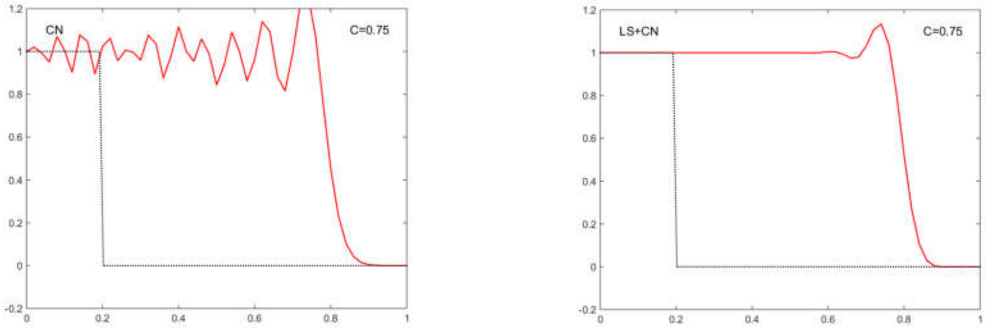

Figure 1: Result for Crank – Nicholson in time and Galerkin in space discretization (Left) and Crank – Nicholson in time and least-squares formulation in space (right).

The results at time *t* = 0.6 are displayed in Figure 1 together with the initial data. They were obtained (for a Courant number *C* = 0.75) by combining the Crank—Nicolson scheme (with linear elements) and the Galerkin formulation, and the least-squares formulation. Note that Crank—Nicolson with least-squares succeeds in removing the spurious oscillations induced by the Galerkin formulation over the whole computational domain. Since Crank—Nicolson is not a monotone scheme, residual oscillations remain at the front. These could be removed using nonlinear viscosity, which is added at the front to render the scheme locally first-order accurate.

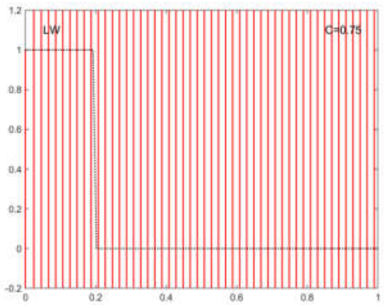

Figure 2: Lax-Wendroff (left), Lax – Wendroff two steps.

The solution obtained by using Lax – Wendroff showed spurious oscillations as expected this instability appears due to the Courant number (C) is equal to 0.75, while the limit of stability of this method is C < 0. 577. These spurious oscillations were induced by the Galerkin formulation. In order to avoid this problem, will be necessary to use a high order method.