**Finite Elements in Fluids** **March 20, 2019**
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**Assignment V:**

**Burgers' equation**
The followin is the inviscid Burgers' equation:

$$\begin{cases} u_t + \left(\frac{u^2}{2}\right)_x = 0 \\ u(x,0) = u_0(x) \end{cases} \qquad \text{or in convective form: } \begin{cases} u_t + u\ u_x = 0 \\ u(x,0) = u_0(x) \end{cases} \tag{1}$$

The correct solution can be determined by the vanishing viscosity approach. The physically correct weak solution of the inviscid Burgers' equation corresponds to the solution of Burgers' equation when viscosity goes to zero.

$$u_t^\epsilon + u^\epsilon u_x^\epsilon = \epsilon u_{xx}^\epsilon \tag{2}$$

## 1 FEM discretization

$$u(x,t) \approx u_h(x,t) = \sum_j N_j(x) u_j(t) \tag{3}$$

Considering $w(x) = N_i(x)$ Seeing it as a matricial system:

$$\boxed{M\dot{U} + C(U)U + \epsilon K U = 0} \tag{4}$$

1. Forward Euler

$$M\frac{U^{n+1} - U^n}{\Delta t} + C(U^n)U^n + \epsilon K U^n = 0$$

$$\boxed{MU^{n+1} = (M - \Delta t(C(U^n) + \epsilon K))U^n = 0} \quad \longrightarrow \text{LINEAR SYSTEM}$$

2. Backward Euler

$$M\frac{U^{n+1} - U^n}{\Delta t} + C(U^{n+1})U^{n+1} + \epsilon K U^{n+1} = 0$$

$$\boxed{MU^n = (M + \Delta t(C(U^{n+1}) + \epsilon K))U^{n+1} = 0} \quad \longrightarrow \text{NON-LINEAR SYSTEM}$$

## 2 Implementation of the Newton-Raphson method.

At each time step, we have to solve $f(U^{n+1}) = 0$ with:

$$f(U) = (M + \Delta t C(U) + \epsilon \Delta t K)U - MU^n$$

Using the Newton-Raphson method we have an initial guess equal to the solution at the previous time-step

$$^0U^{n+1} = U^n$$

And then, we have to iterate (k) until convergence:

$$^{k+1}U^{n+1} =^k U^{n+1} - J^{-1}(^kU^{n+1})f(^kU^{n+1})$$

where $J = \dfrac{df}{dU}$ is the Jacobian matrix.

The Jacobian for this particular case is the following one:

$$J(U) = \frac{df}{dU} = M + 2\Delta t C(U) + \epsilon \Delta t K$$

Therefore, the calculation of the matrix C (U) should be done in each iteration k.

The solution of the Burgers' equation in the domain $[a, b]$ for the three different methods are shown in the following figure. The final time is $tf = 4$, the space-time discretization with $h = 0,02$ and $t = 0,005s$.

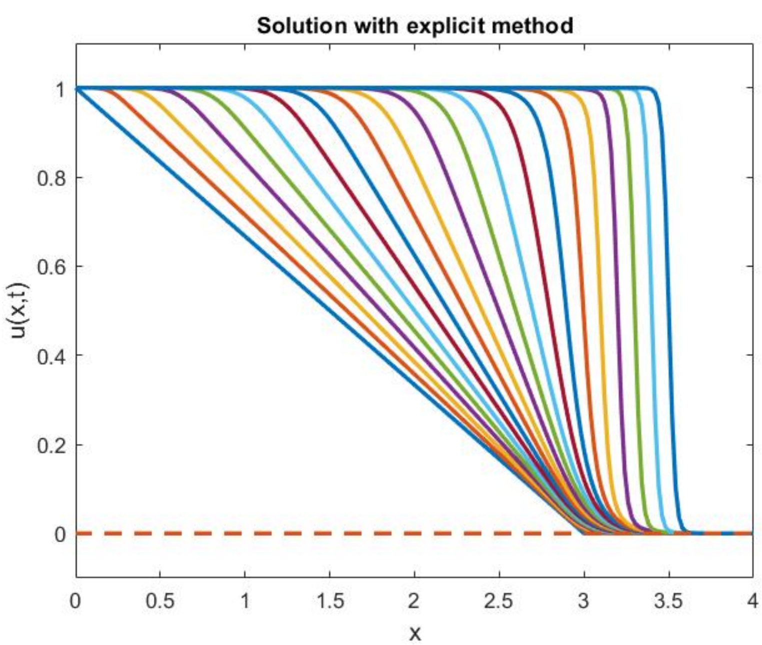


Figure 2.1: Forward Euler

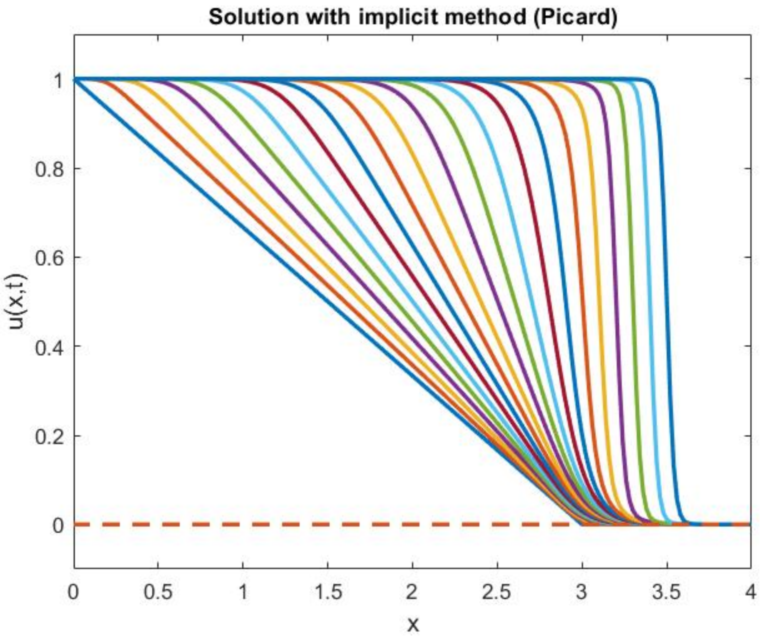


Figure 2.2: Picard's Method

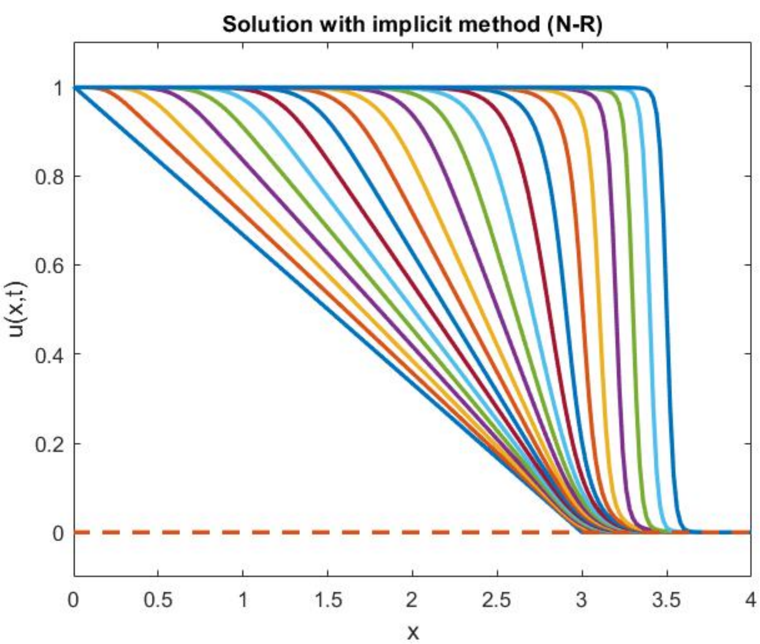


Figure 2.3: Newton-Raphson Method

Finally a last graph is presented where a comparison of the three methods can be observed. In it, it can be seen how they behave in a similar way but it is relevant to note that the Newton-Raphson method has a quadratic convergence although it has some disadvantages such as the matrix J(x) may be singular for some x, the computational cost: at every iteration, compute matrix J(x) and vector f(x) and solve linear system.

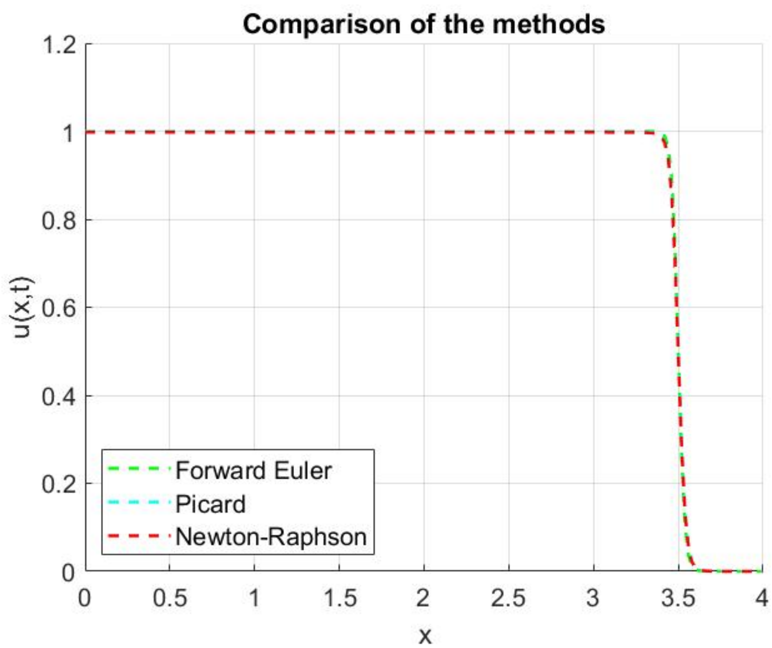


Figure 2.4: Comparison of all three methods.