Finite Elements in Fluids **March 27, 2019**
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Assignment VI:

1 Derive the system of equations for the stress-based Stokes problem

In differential form, a steady Stokes problem is stated as follows in terms of Cauchy stress: given the body force b, prescribed velocities v_D on portion Γ_D of the boundary and imposed boundary tractions t on the remaining portion Γ_N, determine the velocity field v and the pressure field p such that:

$$-\nabla \cdot \sigma = b \quad \text{in } \Omega \qquad \text{(equilibrium)} \tag{1}$$
$$\nabla \cdot v = 0 \quad \text{in } \Omega \qquad \text{(incompressibility)} \tag{2}$$
$$v = v_D \quad \text{on } \Gamma_D \qquad \text{(Dirichlet b.c.)} \tag{3}$$
$$n \cdot \sigma = t \quad \text{on } \Gamma_N \qquad \text{(Neumann b.c.)} \tag{4}$$

The weak form can be obtained multiplying the equation of motion (1) by the velocity test function w and integrating by parts the stress term, thereby generating the natural boundary condition (4) on Γ_N. Similarly, the incompressibility condition (2) is multiplied by the pressure test function q and the result integrated over the computational domain Ω. Thus the weak form of Stokes problem becomes: given b, v_D and the boundary traction t, find the velocity field $v \; \epsilon \; \mathcal{S}$ and the pressure field $p \; \epsilon \; \mathcal{Q}$, such that for all velocity test functions $w \; \epsilon \; \mathcal{V}$ and all pressure test functions $q \; \epsilon \; \mathcal{Q}$,

$$\begin{cases} -\int_\Omega w \cdot (\nabla \cdot \sigma) d\Omega = \int_\Omega w \cdot b \; d\Omega \\ \\ \int_\Omega q \nabla \cdot v \; d\Omega = 0 \end{cases} \tag{5}$$

Using the divergence theorem:

$$\int_\Omega \nabla w \; : \; \sigma \; d\Omega - \int_{\Gamma_D \cup \Gamma_N} n \cdot \sigma \cdot w \; d\Gamma = \int_\Omega w \cdot b \; d\Omega$$

And we know from the boundary conditions that:

1. w=0 on Γ_D
2. $n \cdot \sigma = t$ on Γ_N

Thus:

$$\int_\Omega \nabla w \ : \ \sigma \ d\Omega = \int_{\Gamma_N} t \cdot w \ d\Gamma + \int_\Omega w \cdot b \ d\Omega$$

If we now consider that for a Newtonian Fluid $\sigma = -pI + \tau$

$$\int_\Omega \nabla w \ : \ (-pI + \tau) \ d\Omega = \int_{\Gamma_N} t \cdot w \ d\Gamma + \int_\Omega w \cdot b \ d\Omega$$

$$-\int_\Omega p \nabla \cdot w \ d\Omega + \int_\Omega \nabla w \ : \ \tau \ d\Omega = \int_{\Gamma_N} t \cdot w \ d\Gamma + \int_\Omega w \cdot b \ d\Omega$$

And finally we have:

$$\begin{cases} -\displaystyle\int_\Omega p \nabla \cdot w \ d\Omega + \int_\Omega \nabla w \ : \ \tau \ d\Omega = \int_{\Gamma_N} t \cdot w \ d\Gamma + \int_\Omega w \cdot b \ d\Omega \\ \displaystyle\int_\Omega q \nabla \cdot v \ d\Omega = 0 \end{cases} \tag{6}$$

If we now replace the deviatoric tensor by:

$$\tau = \lambda\cancel{(\nabla \cdot v)}I + 2\mu \nabla^s v$$

Where:

$$[\nabla^s v]_{ij} = \frac{1}{2}\Big(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\Big)$$

Finally,

$$\boxed{\begin{cases} -\displaystyle\int_\Omega p \nabla \cdot w \ d\Omega + 2\mu \int_\Omega \nabla w \ : \ \nabla^s v \ d\Omega = \int_{\Gamma_N} t \cdot w \ d\Gamma + \int_\Omega w \cdot b \ d\Omega \\ \displaystyle\int_\Omega q \nabla \cdot v \ d\Omega = 0 \end{cases}} \tag{7}$$

Velocity and pressure approximations

$$v^h(x) = \sum_{j=1}^{n} v_j N_j(x) = \sum_{j=1}^{n} \begin{bmatrix} v_x^j \\ v_y^j \end{bmatrix} N_j(x)$$

$$= \begin{bmatrix} N_1 & 0 & N_2 & 0 & \cdots & N_n & 0 \\ 0 & N_1 & 0 & N_2 & \cdots & 0 & N_n \end{bmatrix} \begin{bmatrix} v_x^1 \\ v_y^1 \\ v_x^2 \\ v_y^2 \\ \vdots \\ v_x^n \\ v_y^n \end{bmatrix} = [\text{mat N}]\text{v} \tag{8}$$

$$p^h(x) = \sum_{j=1}^{m} p_j \hat{N}_j(x) = \begin{bmatrix} \hat{N}_1 & \hat{N}_2 & \cdots & \hat{N}_m \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ v_x^2 \\ \vdots \\ p_m \end{bmatrix} = \hat{N} p \tag{9}$$

The symmetric tensor $\nabla^s v$ is called the rate of deformation (or strain rate) tensor and it is defined as follows:

$$v_{ij} := \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)$$

We can now use the next property:

$$\nabla^s v = \frac{\nabla v + (\nabla v)^T}{2}$$

Thus, the first equation of the weak form can be rewritten as:

$$-\int_\Omega p \nabla \cdot w \, d\Omega + \mu\Big(\int_\Omega \nabla w \ : \ \nabla v \, d\Omega + \int_\Omega \nabla w \ : \ (\nabla v)^T \, d\Omega\Big) = \int_{\Gamma_N} t \cdot w \, d\Gamma + \int_\Omega w \cdot b \, d\Omega$$

Finally,

$$\nabla w : \nabla v = \frac{\partial w_x}{\partial x}\frac{\partial u_x}{\partial x} + \frac{\partial w_y}{\partial x}\frac{\partial u_y}{\partial x} + \frac{\partial w_x}{\partial y}\frac{\partial u_x}{\partial y} + \frac{\partial w_y}{\partial y}\frac{\partial u_y}{\partial y}$$

$$
= \begin{bmatrix} \frac{\partial w_x}{\partial x} & \frac{\partial w_y}{\partial x} & \frac{\partial w_x}{\partial y} & \frac{\partial w_y}{\partial y} \end{bmatrix} \begin{bmatrix} \frac{\partial u_x}{\partial x} \\ \frac{\partial u_y}{\partial x} \\ \frac{\partial u_x}{\partial y} \\ \frac{\partial u_y}{\partial y} \end{bmatrix} = g(w) \cdot g(v) \tag{10}
$$

And,

$$
\nabla w : (\nabla v)^T = \frac{\partial w_x}{\partial x}\frac{\partial u_x}{\partial x} + \frac{\partial w_y}{\partial x}\frac{\partial u_x}{\partial y} + \frac{\partial w_x}{\partial y}\frac{\partial u_y}{\partial x} + \frac{\partial w_y}{\partial y}\frac{\partial u_y}{\partial y}
$$

$$
= \begin{bmatrix} \frac{\partial w_x}{\partial x} & \frac{\partial w_y}{\partial x} & \frac{\partial w_x}{\partial y} & \frac{\partial w_y}{\partial y} \end{bmatrix} \begin{bmatrix} \frac{\partial u_x}{\partial x} \\ \frac{\partial u_x}{\partial y} \\ \frac{\partial u_y}{\partial x} \\ \frac{\partial u_y}{\partial y} \end{bmatrix} = g(w) \cdot h(v) \tag{11}
$$

Velocity gradient and divergence approximations

$$
g(v^h) = \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \cdots & \frac{\partial N_n}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & \cdots & 0 & \frac{\partial N_n}{\partial x} \\ \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \cdots & \frac{\partial N_n}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & \cdots & 0 & \frac{\partial N_n}{\partial x} \end{bmatrix} \begin{bmatrix} v_x^1 \\ v_y^1 \\ v_x^2 \\ v_y^2 \\ \cdots \\ v_x^n \\ v_y^n \end{bmatrix} = [gradN]v \tag{12}
$$

Then we can substitute on the weak form and express it in matrix form:

$$
\begin{bmatrix} \mu\Big(K + K^T) & G^T \\ -G & 0 \end{bmatrix} \begin{bmatrix} v \\ p \end{bmatrix} = \begin{bmatrix} b + b_N \\ 0 \end{bmatrix} \tag{13}
$$

2 Linearize the matrix C(U) for the Newton-Raphson method

Navier-Stokes non-linear matricial system:

$$r = \begin{bmatrix} K + C(v) & G^T \\ G & 0 \end{bmatrix} \begin{bmatrix} V \\ P \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix} \tag{14}$$

If we want to solve the system of equations we can apply Newton-Raphson method:

$$r = \begin{bmatrix} (K + C(v))v + G^T p - f \\ Gv \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Thus, the solution can be found by iterating:

$$\Big\{ J(x^k)\Delta x^{k+1} = -r(x^k) x^{k+1} = x^k + \Delta x^{k+1}$$

Where J(x) is the Jacobian and it can be defined as it follows:

$$J(x) = \begin{pmatrix} \dfrac{\partial r_1}{\partial v} & \dfrac{\partial r_1}{\partial p} \\ \dfrac{\partial r_2}{\partial v} & \dfrac{\partial r_2}{\partial p} \end{pmatrix}$$

We need now to focus on the term $\frac{\partial r_1}{\partial v}$ we will use the bilinear operation:

$$c(w, v, v) = \int_\Omega w \cdot (v \cdot \nabla) v \; d\Omega$$

After the dicretization and Galerkin application we can write it in matrcial form as:

$$[C(v^h)]_{ij} = \Big(N_i, \Big(\Big(\sum_{j=1}^{n} v_j N_j(x) \Big) \cdot \nabla \Big), v^h \Big)$$

Then c(w,v,v) can be computed as:

$$[Z(v)]_{ij} = \Big(N_i, \Big(V \cdot \nabla \Big), N_j \Big) + \Big(N_i, \Big(N_j \cdot \nabla \Big), v \Big)$$

And finally;

$$J(x) = \begin{bmatrix} K + Z & G^T \\ G & 0 \end{bmatrix} \tag{15}$$

Where $Z(v)_{ij}$ will be evaluated at each iteration k.