Finite Elements in Fluids **April 3, 2019**
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Assignment VII: Stokes problem.

1 For the unstable elements, program a stabilization method.

Stokes equation:

$$-\nu\nabla^2 v + \nabla p = b \qquad in\Omega\nabla \cdot v = 0 \qquad in\Omega$$

Weak form:

$$\int_\Omega \nabla w : \nu\nabla v \, d\Omega - \int_\Omega p\nabla \cdot w \, d\Omega = \int_\Omega w \cdot b \, d\Omega$$
$$\int_\Omega q\nabla \cdot v \, d\Omega$$

And finally the Galerkin discretization in matrix form:

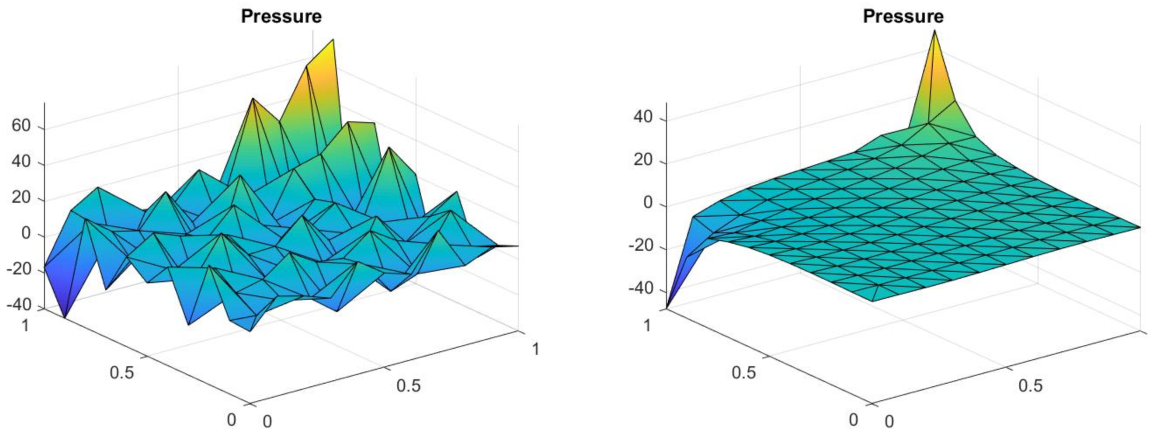
$$\begin{bmatrix} K & G \\ G^T & 0 \end{bmatrix} \begin{bmatrix} u \\ p \end{bmatrix} = \begin{bmatrix} f \\ h \end{bmatrix} \tag{1}$$

1.1 Solve the problem using different element types:

We have the next element types:

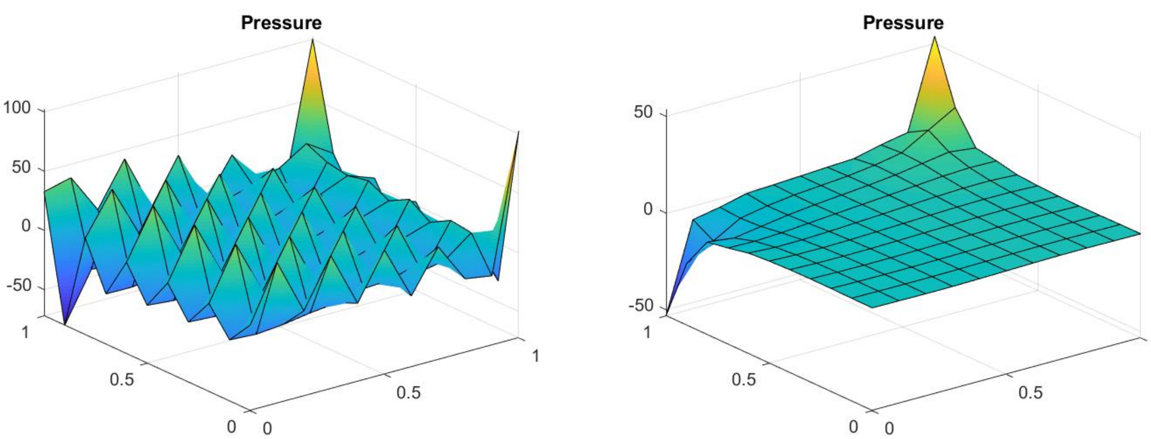
- P1P1(linear/linear triangle),
- P2P1 (quadratic/linear triangle),
- Q1Q1(linear/linear quadrilateral),
- Q2Q1 (quadratic/linear quadrilateral).

In the image below the results obtained for each of the elements are shown without taking into account any type of stabilization.



(a) P1P1 (b) P2P1

Figure 1.1



(a) Q1Q1 (b) Q2Q1

Figure 1.2

After these first results, it is possible to see how the elements used in the first and third images are not stable, this is because these elements do not satisfy the LBB condition.

1.2 Stabilization method.

The method consists of modifying the variational form of the Stokes problem by the addition of the terms emanating from the minimization of the least-squares form.

Stabilization term

$$\sum_e \int_\Omega \tau \mathcal{L}(w,q) \cdot \Big(\mathcal{L}(v,p) - \mathcal{F} \Big) d\Omega \tag{2}$$

The discretization of the stabilized weak form yields:

$$\begin{pmatrix} K + \overline{K} & G^T + \overline{G}^T \\ -G + \overline{G} & 0 + \overline{L} \end{pmatrix} \begin{pmatrix} v \\ p \end{pmatrix} = \begin{pmatrix} f + \overline{f}_w \\ 0 + \overline{f}_q \end{pmatrix} \tag{3}$$

In this particular case, the only elements that need stabilization are the linear ones so we can eliminate the terms with second derivatives, thus:

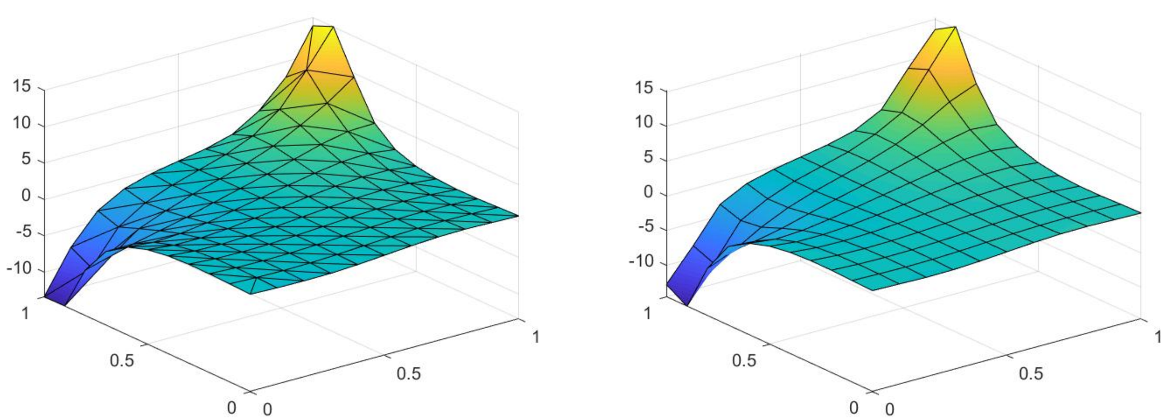
$$\begin{pmatrix} K & G^T \\ -G & \overline{L} \end{pmatrix} \begin{pmatrix} v \\ p \end{pmatrix} = \begin{pmatrix} f \\ \overline{f}_q \end{pmatrix} \tag{4}$$

Where:

$$\overline{L} = \sum_e \int_{\Omega_e} \tau_1 (\nabla q) \cdot (\nabla p) d\Omega \tag{5}$$

$$\overline{f}_q = \sum_e \int_{\Omega_e} \tau_1 (\nabla q) \cdot (-f) d\Omega \tag{6}$$

After adding this terms the solution for P1P1 and Q1Q1 are stables:



(a) Q1Q1 (b) Q2Q1

Figure 1.3