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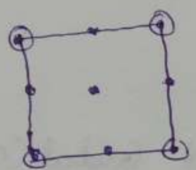
FEF - Final Exam

Problem-1

Stokes problem;

$$\begin{cases} -\nu \nabla^2 v + \nabla p = b & \text{in } \Omega \\ \nabla \cdot v = 0 & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega \end{cases} \quad \text{with } Q_2Q_1 \text{ FE.}$$

a) Yes, $Q_2Q_1$ pair of element spaces are suitable to discretise the equation above.



$(Q_2Q_1)$

It provides biquadratic approximation for velocity and bi-linear approximation for pressure.

It also satisfies LBB condition.

b) Yes, GLS formulation is suitable for the Stokes equations. ~~Even~~ For the elements with equal order interpolations, GLS stabilization turns the unstable Galerkin formulation into a stable one.

c) The pressure is calculated only at the red lines ( between the two element edges). Only unknowns of Global problem are solved on edges.

$$\begin{bmatrix} \hat{K} & \hat{G} \\ \hat{G}^T & 0 \end{bmatrix} \begin{Bmatrix} \hat{u} \\ p \end{Bmatrix} = \begin{Bmatrix} \hat{f}_{\hat{u}} \\ \hat{f}_p \end{Bmatrix}$$

Related to pressure approximation.

~~The~~ Pressure is the average value for the whole element

d) The size of the HDG problem depends on order of approximation.

For the Local problem it is P order and

For the Global problem it is of the order P+1

e) Implementation of an HDG solver for stokes equation

step1: Breaking down the problem into Local and Global (as in D.G)

step2: Identifying the transmission conditions

step3: Introducing a hybrid variable $\hat{u}$, which will be common in both Global and Local problem elements.

step4: obtaining the weak formulation

step5: Exploit the definition of the numerical fluxes and integrate by parts again. we will get the discrete weak problem

step6: Element by Element solving and determine $u_i(\hat{u})$ & $q_i(\hat{u})$ as functions of $\hat{u}$.

step8: Solve for $\hat{u}$.

Step9: HDG post processing

Problem-2

$$\begin{cases} \frac{\partial u}{\partial t} - \nu\nabla^2 u + (u.\nabla)u + \sigma u + \nabla p = f & \text{in } \Omega, t>0 \\ \nabla . u = 0 & \text{in } \Omega, t>0 \\ u = 0 & \text{on } \partial\Omega, t>0 \\ u = u_D & \text{in } \Omega, t=0 \end{cases}$$

with Q1Q1 element library only.

(a) Time discrete problem using the Crank-Nicolson method.

$$\frac{u^{n+1} - u^n}{\Delta t} + C(u) + K(u)^{n+\theta} + \sigma u^{n+\theta} + \nabla p^{n+\theta} = f^{n+\theta}$$

$$\nabla . u^{n+\theta} = 0$$

with, $C(u) = (u.\nabla)u$

$K(u) = -\nu\nabla^2 u$

$\Delta t = t^{n+1} - t^n$

$f^{n+\theta} = \theta f^{n+1} + (1-\theta) f^n$ , $f^n \approx f(t^n)$

$\theta = 1/2$ - Crank-Nicolson ~~is unconditionally stable method~~

(b) Weak form can be written using WRM and after some algebra

$$\int_\Omega w.u_t \, d\Omega + \int_\Omega (\nabla w):(\nabla\nabla u)\, d\Omega + \int_\Omega w.(a.\nabla)u \, d\Omega + \int_\Omega \sigma u \, d\Omega - \int_\Omega (\nabla . w) \, d\Omega = \int_\Omega w.f \, d\Omega$$

$$\int_\Omega q \nabla . u \, d\Omega = 0$$

where, w and q represents weighting functions for equilibrium condition and incompressibility condition respectively.

(c) Finite element discretization yields following form of the previous equation.

$$M u_t + [K + C(u(t))]u(t) + Gp(t) = f(t, u(t))$$

$$G^T u(t) = h(t)$$

$$u(0) = u_0 - u_D$$

where, M - consistent mass matrix

C - convective matrix

K - viscosity matrix

Combining with Time discretisation

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \theta \left[ \frac{f - [K + C(u)]u^{n+1} - Gp^{n+1}}{M} - \frac{f - [K + C(u)]u^n - Gp^n}{M} \right] + \frac{f - (K + C(u))u^n - Gp^n}{M}$$

ie, $$\frac{\Delta u}{\Delta t} M + \theta \left[ (K + C(u))\Delta u + G\Delta p \right] = f - [K + C(u)]u^n - Gp^n$$

The algebraic equation can be approximated as below

$$\begin{bmatrix} M + \theta\Delta t(K + C(u)) & \Delta t G^T \\ G & 0 \end{bmatrix} \begin{bmatrix} \Delta u \\ \Delta P \end{bmatrix} = \begin{bmatrix} \Delta t[f - (K + C(u))u^n - Gp^n] \\ 0 \end{bmatrix}$$

Q1Q1,  does not satisfy LBB condition to solve the above.

Stabilization would be helpful.

d) It would be better to use Picard's method to solve this non-linear system of equations.

Picard's Algorithm for $A(x)x = b(x)$

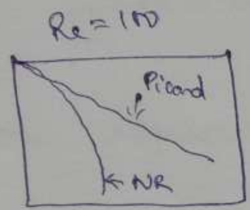
$x = A(x)^{-1} b(x)$

$\Delta x^{k+1} = x^{k+1} - x^{k}$ → $\underline{\Delta x^{k+1} = A(x^{k})^{-1}\left(b(x^{k}) - A(x^{k})x^{k}\right)}$

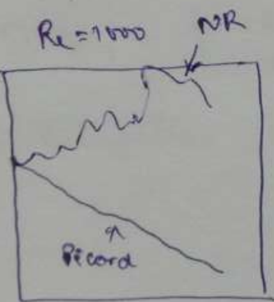
$x^{k+1} = A(x^{k})^{-1} b(x^{k})$

Picard's method is more robust than the N-R method of iterations.

e)




-> Here, the methods are behaving as expected
-> Convergence is linear in the case of Picard
-> N-R method converges quadratically
-> This observation is for Re = 100.




-> This graph shows the expected behavior.
-> Picard method still has linear convergence
-> Newton method is not able to converge the solution for $R_e = 1000$.
-> For higher values of $R_e$ the convective term dominates diffusion and hence NR method can fail to converge.