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Assignatura: FEF Resit

Cognoms

Nom

Pàgina ___ de ___

DNI

Master in Computational Mechanics

Ex-1

$$-\nu u_{xx} + \beta u_x = 0 \quad , \quad x \in (0,1)$$

$$u = 0 \quad , \quad x = 0$$

$$u = 1 \quad , \quad x = 1$$

Dirichlet B.C

(a) $S := \{ u \in \mathcal{H}^1(\Omega) \mid u = 0 \text{ on } x = 0 \ \& \ u = 1 \text{ on } x = 1 \}$

$V := \{ w \in \mathcal{H}^1(\Omega) \mid w = 0 \text{ on } \Gamma_D \}$

$$-\nabla \cdot \nu \nabla u + \beta \cdot \nabla u = 0$$

$$-\int_\Omega w \, \nabla \cdot (\nu \nabla u) \, d\Omega + \int_\Omega w \, (\beta \cdot \nabla u) \, d\Omega = 0$$

using $w = 0$ on Γ_D and integrating by parts

$$\int \nabla w \cdot (\nu \nabla u) \, d\Omega + \int w \, (\beta \cdot \nabla u) \, d\Omega = 0$$

$$\Rightarrow \boxed{a(w, u) + c(\beta, w, u) = 0}$$

(b) $a^h(w^h, u^h) + c(\beta, w^h, u^h) = 0 \quad \forall w^h \in V^h$

$$u^h(x) = \sum_{B \in \eta \setminus \eta_D} N_B(x) \, u_B + \sum_{B \in \eta_D} N_B(x) \, u_D(x_B)$$

$w^h \in V^h$

Discrete form

$$\sum_{B\in n\setminus n_D} [a(N_A, N_B) + c(\beta; N_A, N_B)] u_B = -\sum_{B\in n_D} [a(N_A, N_B) + c(\beta; N_A, N_B)] u_D(x_B)$$

~~Now input the B.C~~

$$\sum_{B\in n\setminus n_D} [a(N_A, N_B) + c(\beta, N_A, N_B)] u_B = -[a(N_A, N_B) + c(\beta; N_A N_B)] u_D(x_B)$$

~~($u = 1$ at $x = 1$)~~
~~($u = 0$ at $x = 0$)~~

Convection Matrix

$$(C + K) u = f$$

$$C = A^e c^e \qquad c^e_{ab} = \int N_a (\beta \cdot \nabla N_b) \, d\Omega$$

$$K = A^e k^e \qquad k^e_{ab} = \int \nabla N_a \cdot \nu \nabla N_b \, d\Omega$$

Diffusion Matrix

$$f^e_a = - \sum_{b=1}^{n_{en}} [a(N_a, N_b)_{\Omega^e} + c(\beta; N_a, N_b)_{\Omega^e}] u^e_{D_b}$$

For linear elements

$$c^e = \beta \int_{\Omega^e} \begin{pmatrix} N_1 \frac{\partial N_1}{\partial x} & N_1 \frac{\partial N_2}{\partial x} \\ N_2 \frac{\partial N_1}{\partial x} & N_2 \frac{\partial N_2}{\partial x} \end{pmatrix} dx = \frac{\beta}{2} \begin{pmatrix} -1 & +1 \\ -1 & +1 \end{pmatrix}$$

$$k^e = \nu \int_{\Omega^e} \begin{pmatrix} \frac{\partial N_1}{\partial x} \frac{\partial N_1}{\partial x} & \frac{\partial N_1}{\partial x} \frac{\partial N_2}{\partial x} \\ \frac{\partial N_2}{\partial x} \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} \frac{\partial N_2}{\partial x} \end{pmatrix} dx = \frac{\nu}{h} \begin{pmatrix} +1 & -1 \\ -1 & +1 \end{pmatrix}$$

$$f = [f_1 \overset{=0}{} \quad f_2 \overset{=0}{} \; - - - - - \; f_{13} \overset{=0}{} + \alpha 1]_{13\times 1}$$

(C) $Pe = \frac{\beta h}{2\nu}$ $h = \frac{1}{12}$

$$= \frac{\cancel{2} \times \frac{1}{\cancel{12}\,6}}{2 \times 0.05} = \frac{1}{6 \times 0.1} = \frac{10}{6} > 1$$

$Pe > 1 \Rightarrow$ It will lead to spurious oscillations
∴ The galerkin formulation becomes inaccurate when convection term dominates diffusion term.

(d) SUPG stabilisation

Residual

$$R(u) = (\beta \cdot \nabla u - \nabla \cdot (\nu \nabla u))\, u$$

For SUPG $P(w) = a \cdot \nabla w$

$$a(w, u) + c(\beta, w, u) + \sum_e \int P(w)\, \tau\, R(u)\, d\Omega = 0$$

$$a(w^h, u^h) + c(\beta, w^h, u^h) + \sum_e \int (\beta \cdot \nabla w^h)\, \tau \left[\beta \cdot \nabla u^h - \nabla \cdot (\nu \nabla u^h)\right] d\Omega = 0$$

Since 1-D conv-diffusion and for linear elements stabilisation term reduces to

$$\boxed{\sum_e \int (\beta \cdot \nabla w^h)\, \tau\, (\beta \cdot \nabla u^h)} \quad ; \tau = \nu / \|a\|^2$$

$$\Rightarrow a(w^h, u^h) + c(\beta, w^h, u^h) + \sum_e \int (\beta \cdot \nabla w^h)\, \tau (\beta \cdot \nabla u^h)\, d\Omega = 0$$

$$\sum_{B \in n|n_0} \left[a(N_a, N_b) + c(\beta, N_a, N_b) \right] u_B + \sum_e \int (\beta \cdot \nabla N_A)\, \tau (\beta \cdot \nabla N_B) u_B \, d\Omega = 0$$

(e) No, the results are not similar to (b) because with ~~the~~ SUPG stabilisation system for $\boxed{Pe > 1}$ will be stabilised.

Ex-2
(a)

$$-(\nabla\cdot(\nu\nabla u) - \nabla\cdot p\,I_{nsd} = b \quad ; \text{ in } \Omega \quad -2(a)$$

$$\nabla\cdot u = 0 \quad \text{in } \Omega \quad -2(b)$$

$$u = u_D \quad \text{on } \Gamma_D \quad -2(c)$$

$$(\nu\nabla u - p\,I_{nsd})\,n = g \quad \text{on } \Gamma_N \quad -2(d)$$

Introducing weight funct^n in 2(a)

$$-\int_{\Omega} w\,\nabla\cdot(\nu\nabla u)\,d\Omega + \int_{\Omega} w\,(\nabla\cdot p\,I_{nsd})\,d\Omega = \int_{\Omega} w\,b\,d\Omega$$

Integrating by parts and by divergence thm.

$$\int_\Omega \nabla w : \nu\,\nabla u\,d\Omega - \int_\Omega (\nabla\cdot w)\,p\,d\Omega = \int_\Omega w\cdot b\,d\Omega + \int_{\Gamma_N} w\cdot g\,d\Gamma$$

$$\left.\begin{aligned} a(w,u) - b(w,p) = (w,f) \quad \forall w \in \mathcal{V} \\ + (w,g)_{\Gamma_N} \end{aligned}\right] - 2\cdot 1$$

similarly introducing weights funcn and integrating for eq 2(b)

$$\mathcal{Q} = p \in L_2(\Omega) \quad ; \; q \in \mathcal{Q}$$

$$\int q\,\nabla\cdot u = 0$$

$$b(u,q) = 0 \quad] - 2\cdot 2$$

$$\left[\begin{aligned} &S : \{v \in H^1(\Omega) \mid u = u_D \text{ on } \Gamma_D\} \quad (\text{trial sol}^n) \\ &\mathcal{V} : H^1_{\Gamma_D}(\Omega) = \{w \in H^1(\Omega) \mid w = 0 \text{ on } \Gamma_D\} \quad (\text{weighting funct}^n) \end{aligned}\right.$$

Eq. 2.1 and 2.2 are the weak forms

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(b) Since, P^2 ~~pol~~ funct^n for velocity and pressure.
$\Rightarrow$ $Q_2 Q_2$ element, which is unstable because it doesn't satisfy LBB condition

(c) P^2 polynomial funct^n of vel., pressure, grad. velocity degree, $k = 2$

$\Rightarrow$ Optimal order of convergence for velocity $= k+1 = 3$

" " " " " ~~pressure~~

grad of velocity $= k+1 = 3$

$\therefore$ convergence will be much faster than that of 1st order.

Numerical solution will be stable

(d) First we will discritize weak form of Stokes problem obtained in 2(a), then we will write discritize form using GLS.

vel comp. are approximated as

$$\left.\begin{aligned} u_i^h(x) &= \sum_{B\in\eta\setminus\eta_{Di}} N_B(x)\, u_{iB} \\ u_{Di}^h(x) &= \sum_{B\in\eta_{Di}} N_B(x)\, u_{Di}(x_B) \end{aligned}\right\} \quad u_i^h = v_i^h + u_{Di}^h$$

But vector version of above eq is

$$v^h(x) = \sum v_i^h(x)\, e_i = \sum_{i=1}^{n_{sd}} \sum_{B\in\eta\setminus\eta_{Di}} N_B(x)\, v_{iB}\, e_i$$

$$w^h(x) = \sum_{i=1}^{n_{sd}} w_i^h(x)\, e_i$$

$$p(x) \approx p^h(x) = \sum \hat{N}_{\hat{A}}(x)\, p_{\hat{A}}$$

Discitised form of eq 2·1

$$\sum_{j=1}^{n_{sd}} \left\{ \sum_{B \in \eta \setminus \eta_{Dj}} a(N_A e_i , N_B e_j)\, u_{jB} \right\} + \sum b(N_A e_i , \hat{N}_{\hat{A}})\, p_{\hat{A}}$$

$$= (N_A e_i , b^h) + (N_A e_i , g^h)_{\Gamma_N} - \sum_{j=1}^{n_{sd}} \left\{ \sum_{B \in \eta_{Dj}} a(N_A e_i \cdot N_B e_j)\, u_{Dj} \right\}$$

Discritized form of eq 2·2

$$\sum_{i=1}^{n_{sd}} \left\{ \sum_{B \in \eta \setminus \eta_{Di}} b(N_B e_i , \hat{N}_{\hat{A}})\, u_{iB} \right\} = -\sum_{i=1}^{n_{sd}} \left\{ \sum_{B \in \eta_{Di}} b(N_B e_i \; \hat{N}_{\hat{A}})\, v_{Di} \right\}$$

$$\Rightarrow \begin{pmatrix} K & G \\ G^T & 0 \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} f \\ h \end{pmatrix}$$

$$K = \int (\nabla N_A)(\nabla N_B)\, d\Omega$$

$N_A \rightarrow$ for weight funct^n (w)

$$G = -\int (\nabla N_A)\, \hat{N}_{\hat{A}}\, d\Omega$$

$N_B \rightarrow$ for velocity

$\rightarrow$ Discrete gradient operator

$$G^T = -\int (\nabla N_B)\, \hat{N}_{\hat{A}}\, d\Omega$$

$\rightarrow$ Discrete divergence operator

$$f \longrightarrow (w, b) + (w, g)_{\Gamma_N} - a(w, v_D)$$

$$h \longrightarrow -b(u_D, q)$$

$$e_1 = (1\ 0\ 0)^T \qquad e_2 = (0\ 1\ 0)^T$$

$$e_3 = (0\ 0\ 1)^T$$

Introducing GLS

$$\mathcal{L}(v,p) = \begin{bmatrix} L_1(v,p) \\ L_2(v,p) \end{bmatrix} = \begin{bmatrix} -\nu \nabla^2 v + \nabla p - b \\ \nabla \cdot v \end{bmatrix}$$

$$F = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \begin{bmatrix} f \\ h \end{bmatrix}$$

$$L(v,p) = (-\nu \nabla^2 v + \nabla p - b, -\nu \nabla^2 v + \nabla p - b)$$

for w, q (weight funct^n) ~~for vel and pressure~~

$$(-\nu \nabla^2 w + \nabla q, -\nu \nabla^2 v + \nabla p - b) = 0 \quad \forall (w,q) \in V \times Q$$

stabilisation term are -

$$\Rightarrow \begin{cases} (-\nu \nabla^2 w, -\nu \nabla^2 v + \nabla p - b) = 0 & \forall w \in V \\ (\nabla q, -\nu \nabla^2 v + \nabla p - b) = 0 & \forall q \in Q \end{cases}$$

Adding these stabilisation terms to galerkin weak form

$v^h \in S^h \quad p^h \in Q^h \quad \forall (w^h, q^h) \in V^h \times Q^h$

$$\begin{cases} a(w^h, v^h) + b(w^h, p^h) + \sum \tau_e (-\nu \nabla^2 w^h, -\nu \nabla^2 v^h + \nabla p^h - b^h)_{\Omega_e} \\ \quad = (w^h, b^h) + (w^h, g^h)_{\Gamma_N} \\ b(v^h, q^h) - \sum \tau_e (\nabla q^h, -\nu \nabla^2 v^h + \nabla p^h - b^h)_{\Omega_e} = 0 \end{cases}$$

$(\nabla q^h, \nabla p^h)$ introduces a non-zero diagonal term in partioned matrix

Due to GLS, equal order interpolation in galerkin formulation becomes stable.

$$\sum_{B\in n\setminus n_D} a(N_A, N_B)u_B + \sum_{\hat{B}\in n} b(N_{Aei}, \hat{N}_{\hat{A}})p_{\hat{A}}$$

$$+ \dot{\sum} \tau_e \left(-\nu\nabla^2 N_A, -\nu\nabla^2 N_A + \nabla N_{\hat{A}} - b\right)$$

$$\begin{pmatrix} K+\bar{K} & G+\bar{G} \\ G^T+\bar{G} & 0+\bar{L} \end{pmatrix} = \begin{pmatrix} f+\bar{f} \\ h+\bar{f}q \end{pmatrix}$$

$$\bar{G}^T \rightarrow \sum \int \tau_1 (\nabla q)(-\nu\nabla^2 v)\, d\Omega$$

$$\bar{L} \rightarrow \sum \int \tau (\nabla q)\cdot(\nabla p)\, d\Omega$$

$$\bar{f}w \rightarrow \sum \int \tau(-\nu\nabla^2 w)\cdot(-f)\, d\Omega$$

$$\bar{f}q \rightarrow \sum \int \tau (\nabla q)(-f)\, d\Omega$$

$$\tau_1 = \alpha \frac{h^2}{4\nu}$$

~~2 (e) Since it's stable we don't need GLS due to LBB condition~~

~~(f)~~

(e) It is stable due to LBB we will still be requiring GLS stabilisation

GLS ensures stability

(f) This is stable as LBB condition is satisfied.

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