FINITE ELEMENTS IN FLUIDS

ASSIGNMENT6- PROPAGATION OF STEEP FRONT

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Steep front Problem definition

$$\begin{cases} u_t + a u_x = 0 & x \in (0,1),\ t \in (0, 0.6] \\ u(x,0) = u_0(x) & x \in (0,1) \\ u(0,t) = 1 & t \in (0, 0.6] \end{cases}$$

$$u_0(x) = \begin{cases} 1 & \text{if } x \leq 0.2, \\ 0 & \text{otherwise} \end{cases}$$

$$a = 1, \Delta x = 2 \cdot 10^{-2}, \Delta t = 1.5 \cdot 10^{-2}$$

This initial condition has been implemented in the 1-d convection codes as-

```
elseif problem == 4
    x0 = 0.2;
    ind = find(X <=x0);
    u = zeros(size(X));
    u(ind) = 1;
```

Courant Number

By definition, Courant number, C= a Δt / Δx = 0.75

Crank-Nicolson in time+ Linear Finite Element Galerkin in space

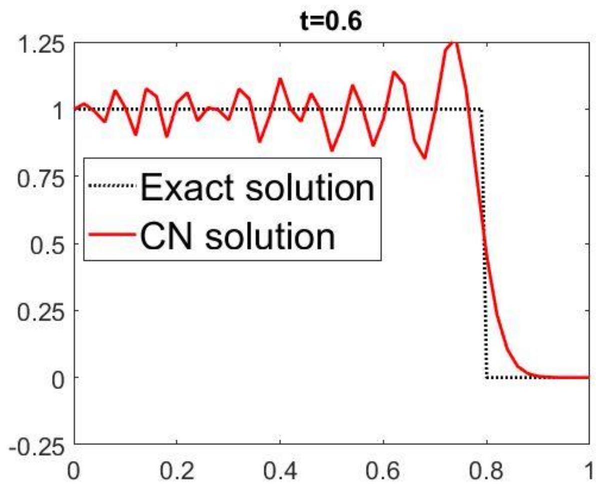
The codes have been implemented by modifying the 1d- convection codes that were given during initial lectures.

The following code was implemented for this problem

```
    case 3 % Crank-Nicolson + Galerkin
        A = M + 1/2*a*dt*C;
        B = -a*dt*C;
        methodName = 'CN';
```

Result for C=0.75

We find that even though Crank Nicolson is implicit method and boasts unconditional stability the solution contains errors.



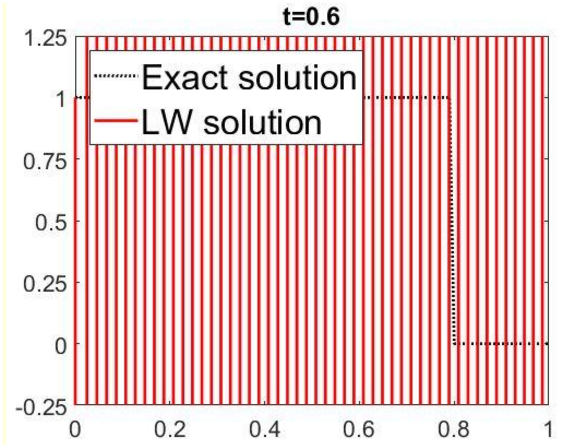
CN+Galerkin

Second Order Lax Wendroff Method

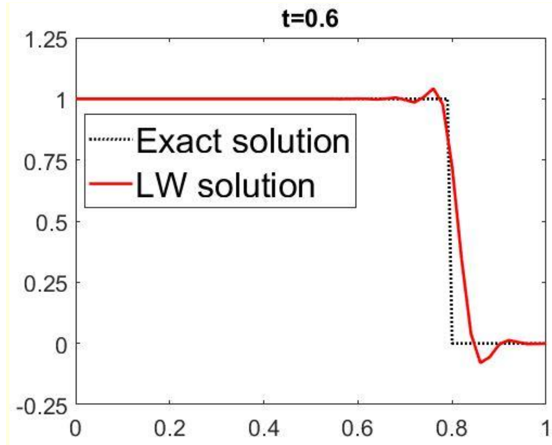
The following code was implemented

```
    case 1 % Lax-Wendroff + Galerkin
        A = M;
        B = -a*dt*C- 0.5*a^2*dt^2*K;
        methodName = 'LW';
```

Results



C=0.75

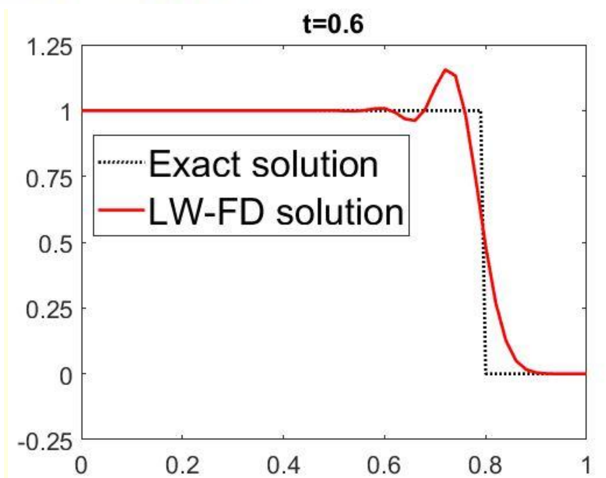


C=0.25

Second order Lax Wendroff is an explicit method with stability criteria $C^2<1/3$. So for C=0.75 , which falls outside it's stability region we get massive oscillations in the solution . However, for C= 0.2, which is well within the stability criteria, we find a relatively stable solution. Some errors still exist near the sharp change of profile but that can be attributed to its second order accuracy.

We can change the original second order LW method to only consider the diagonal elements of the mass matrix forming the Second Order Lax-Wendroff Lumped Mass formulation which gives better stability for C=0.75. The codes and results are given below-

```
    case 2 % Lax-Wendroff with lumped mass matrix + Galerkin
        A = diag(sum(M));
        B = -a*dt*C - 0.5*a^2*dt^2*K;
        methodName = 'LW-FD';
```



C=0.75

Second Order Two-Step Lax Wendroff Method

$$u^{n+1} = u^n + \Delta t\, u_t + \frac{1}{2}\Delta t^2 u_{tt}$$

$$\Rightarrow u^{n+1} = u^n + \Delta t \frac{\partial}{\partial t} + \frac{1}{2}\Delta t^2 \frac{\partial^2}{\partial t^2}$$

$$\Rightarrow u^{n+1} = u^n + \Delta t \frac{\partial}{\partial t} \underbrace{\left(1 + \frac{\Delta t}{2}\frac{\partial}{\partial t}\right)}_{\tilde{u}}$$

Forming the following 2 step equations-

$$\tilde{u} = u^n + \frac{\Delta t}{2} u_t^n$$

$$u^{n+1} = u^n + \Delta t\, \tilde{u}_t$$

After performing a simple galerkin discritization in space we formulate the following code to incorporate the 2 step nature of the method.

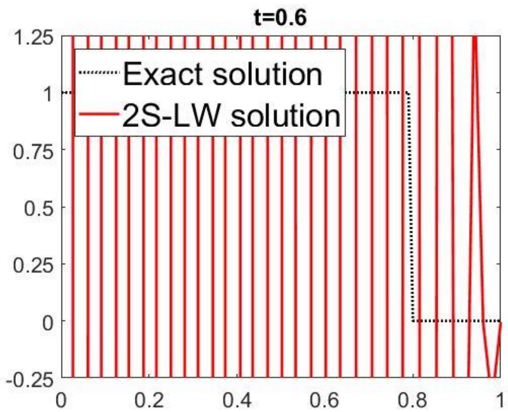
Changes in *system.m*

```
    case 5 %Second Order 2-Step Lax Wendroff + Galerkin
        %First Step
        A = M;
        B = -a*dt*0.5*C;
        methodName = '2S-LW';
    case 6 % Second Step
        A=M;
        B = -a*dt*C;%Coefficient of u_bar in main.m
        methodName = '2S-LW';
```

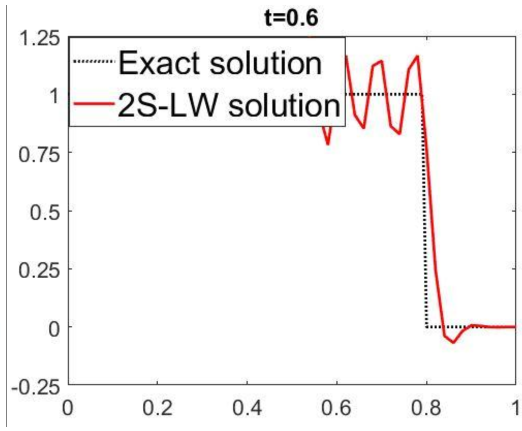
Changes in *main.m*

```
if method==5
    for n=1:nStep
    [A,B,methodName] = System(5,M,K,C,a,dt);
    Du = A\(B*u(1:nPt,n));
    u_bar= u(1:nPt,n) + Du;
    clear A,B ;
      [A,B,methodName] = System(6,M,K,C,a,dt);
    Du = A\(B*u_bar);
    u(1:nPt,n+1) = u(1:nPt,n) + Du;
    end
```

Results



C=0.75



C=0.25

The result obtained for C=0.75 is as expected as it falls outside the stability criteria. However, the solution for C=0.25 also shows some oscillations as the solution approaches the steep front which is unexpected as 2-Step Lax Wendroff is expected to show a level of diffusivity over single stage second order Lax-Wendroff which is missing here.