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Mid term - Finite elements in Fluids

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Exercise 2

given the steady state convection-diffusion equation

$$-\nu \frac{\partial^2 u}{\partial x^2} + \beta \frac{\partial u}{\partial x} = 0 \qquad x \in (-1, 1)$$

$$u(-1) = 0 \qquad \nu = 0.03$$

$$u(1) = -1 \qquad \beta = 1.8$$

a) Write the weak form of the problem & the Galerkin discretized system of equations

Answer:

to find the weak form, multiply by test function $w \in H^1$, integrate in the domain and use integration by parts to lower the continuity requirements (solution u will only need to be C^1, not C^2 as in the strong form. Given:

$$-\nu \frac{\partial^2 u}{\partial x^2} + \beta \frac{\partial u}{\partial x} = 0 \qquad \text{[multiply by } w \text{ and integrate]}$$

$$-\nu \int_{-1}^{1} w \frac{\partial^2 u}{\partial x^2} dx + \beta \int_{-1}^{1} w \frac{\partial u}{\partial x} dx = 0 \qquad (1)$$

(2) use integration by parts on the $\frac{\partial^2 u}{\partial x^2}$ term

$$\int_{-1}^{1} w \frac{\partial^2 u}{\partial x^2} dx = -\int_{-1}^{1} \frac{\partial w}{\partial x}\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial x}\Big|_{-1}^{1}$$

but w has compact support (is zero on the border of the domain) so

$$\int_{-1}^{1} w \frac{\partial u}{\partial x} dx = -\int_{-1}^{1} \frac{\partial w}{\partial x}\frac{\partial u}{\partial x} dx \qquad (2)$$

introducing (2) into (1) renders

$$\mu\int_{-1}^{1} \frac{\partial w}{\partial x}\frac{\partial u}{\partial x} dx + \beta\int_{1}^{1} w\frac{\partial u}{\partial x} dx = 0 \qquad (3)$$

now we use $u_h = \sum_i u_i N_i$ to approximate u

Because we use Galerkin, $w_j = N_j$

(test and basis functions are the same).

so $\frac{\partial u}{\partial x} = \sum_i u_i \frac{dN_i}{dx}$ $\qquad \frac{\partial w_j}{\partial x} = \frac{dN_j}{dx}$

introducing into (3) we get:

$$\mu\int_{-1}^{1} \sum_i u_i \frac{dN_i}{dx} \frac{dN_j}{dx} + \beta\int N_j \sum_i u_i \frac{dN_i}{dx} = 0 \quad (4$$

which is a linear system of equations of the form

$$\bar{\bar{K}}\,\bar{u} = 0$$

3 where $K_{ji} = \left[\nu \frac{dN_i}{dx} \frac{dN_j}{dx} + \beta N_j \frac{dN_i}{dx} \right]$

and $u_j = [u_1, u_2 \dots u_n]$

the Dirichlet border conditions u_0 and u_{n+1} are known

b). given $N = 20$, do you expect oscillations?

Answer: there will be oscillations if the Peclet number is $P_e > 1$

$$P_e = \frac{|a| h}{2\nu}$$ in this case $|a| = \beta = 1.8$

$h = 2/20 = 1/10$

$\nu = 0.03$

so $P_e = \frac{1.8}{2 \times 10 \times 0.03} = \frac{1.8}{0.6}$

$P_e = 3 > 1 \Rightarrow$ the standard

Galerkin will oscillate! as the form reduces to $(u_{i+1} - u_i) = \frac{P_e + 1}{1 - P_e} (u_i - u_{i-1})$

- Write the discrete form of the equation using sub-grid scale (S.G.S.) stabilization

Using compact integral forms, eq (3) is

$$\nu\, a(w, u) + c(\beta, w, u) = 0$$

So the SGS form is

$$\nu\, a(\bar{w}, \bar{u}) + c(\beta, \bar{w}, \bar{u}) + \sum_e \tau \left(-\mathcal{L}^*(\bar{w})\, \mathcal{L}(\bar{u})\right)_{\Omega e} = 0$$

where $\bar{w}$ and $\bar{u}$ are the coarse scale solutions and the operator $\mathcal{L}^*(u) = -\beta \frac{du}{dx} - \nu \frac{\partial^2 u}{\partial x^2}$ and $\mathcal{L}(u) = \beta \frac{du}{dx} - \nu \frac{\partial^2 u}{\partial x^2}$

Since S.G.S. is a stabilization method, like the ones we studied in class (SUPG & GLS) it will provide a smoother solution, without the oscillations observed using Galerkin [results are similar to SUPG or GLS]

c) consistent stabilization

For the convection-diffusion problem, when $Pe > 1$ convection 'overrides' diffusion, leading to oscillation using galerkin techniques. Stabilization consists of artificially adding diffusion to smooth out the solution without having to refine the grid excessively $(h \to 0)$ consistent stabilization methods only add diffusion in the direction of the convection, allowing the removal of oscillations without excessive 'smearing' of the solution

(5) the quantities involved in stabilization are sumarized in the Peclet number

$$P_e = \frac{|a| h}{2\nu}$$

a = convection speed

h = grid size

ν = diffusivity constant.

Exercise 1

a) Write second order T.G. approximation for inviscid Burger's eq. $\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = 0$

<u>Answer</u>: using Burger's eq. we can write

$$\frac{\partial u}{\partial t} = - u \frac{\partial u}{\partial x} \quad (1)$$

$$\frac{\partial^2 u}{\partial t^2} = - \frac{\partial}{\partial t}\left[u \frac{\partial u}{\partial x}\right] \qquad \text{if} \quad f = \frac{u^2}{2}$$

$$= - \frac{\partial}{\partial t}\left[\frac{\partial f}{\partial x}\right] \qquad \left\{ \begin{aligned} \frac{\partial f}{\partial x} &= \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x} \\ &= u \frac{\partial u}{\partial x} \end{aligned} \right.$$

$$= - \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial t}\right]$$

$$= - \frac{\partial}{\partial x}\left[u \frac{\partial u}{\partial t}\right] \qquad \left\{ \begin{aligned} \frac{\partial f}{\partial t} &= \frac{\partial f}{\partial u} \frac{\partial u}{\partial t} \\ &= u \frac{\partial u}{\partial t} \end{aligned} \right.$$

$$= - \frac{\partial}{\partial x}\left[u\left(-u \frac{\partial u}{\partial x}\right)\right] \qquad \left\{ \frac{\partial u}{\partial t} = - u \frac{\partial u}{\partial x} \right.$$

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left[u^2 \frac{\partial u}{\partial x}\right] \quad (2)$$

(6)

using Taylor

$$u^{n+1} = u^n + \Delta t \frac{\partial u}{\partial t}^n + \frac{1}{2} \Delta t^2 \frac{\partial^2 u}{\partial t^2}^n + O(\Delta t^3)$$

We can write

$$\frac{\Delta u}{\Delta t} = \frac{u^{n+1} - u^n}{\Delta t} = \frac{\partial u}{\partial t}^n + \frac{1}{2} \Delta t \frac{\partial^2 u}{\partial t^2}^n + T.E. \quad (3)$$

neglecting the Truncation error and introducing (1) and (2) into (3) we get.

$$\frac{\Delta u}{\Delta t} = -u \frac{\partial u}{\partial x} + \frac{1}{2} \Delta t \frac{\partial}{\partial x}\left[u^2 \frac{\partial u}{\partial x}\right] \quad (4)$$

which is explicit and second order in time

Next we write the weak form

$$\int_0^1 w \frac{\Delta u}{\Delta t}\, dx = -\int_0^1 w \frac{\partial u}{\partial x}\, dx + \frac{\Delta t}{2} \int_0^1 w \frac{\partial}{\partial x}\left[u^2 \frac{\partial u}{\partial x}\right] dx \quad (5$$

applying integration by parts on the second term of the R.H.S we get:

$$\int_0^1 w \frac{\partial}{\partial x}\left[u^2 \frac{\partial u}{\partial x}\right] dx = -\int \frac{\partial w}{\partial x}\left[u^2 \frac{\partial u}{\partial x}\right] dx + w \frac{\partial}{\partial x}\left[u^2 \frac{\partial u}{\partial x}\right]\Bigg|_0^1$$

but $w = 0$ on $x = 0$ and $x = 1$ so

$$\int_0^1 w \frac{\partial}{\partial x}\left[u^2 \frac{\partial u}{\partial x}\right] dx = -\int_0^1 \frac{\partial w}{\partial x}\left[u^2 \frac{\partial u}{\partial x}\right] dx \quad (6)$$

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introducing (6) into (5) we get:

$$\int_0^1 w \frac{\Delta u}{\Delta t}\, dx = -\int_0^1 w \frac{\partial u}{\partial x}\, dx - \frac{\Delta t}{2} \int_0^1 \frac{\partial w}{\partial x}\left[u^2 \frac{\partial u}{\partial x}\right] d$$

which is the requeste T.G. method (one step)

Discretizing using $u_h = \sum_j^i N_j(x)\, u_j(t)$

$$w_i = N_i(x)$$

we obtain a system of this kin.

$$\frac{1}{\Delta t}\int_0^1 w\, u^{n+1}\, dx = \frac{1}{\Delta t}\int_0^1 N_i \sum_j^i N_j u_j\, dx$$

$$\int_0^1 w \frac{\partial u}{\partial x}\, dx = \int_0^1 N_i \sum_j^i \frac{dN_j}{dx} u_j \quad dx$$

$$\int_0^1 \frac{\partial w}{\partial x}\left[u^2 \frac{\partial u}{\partial x}\right] = \int_0^1 N_i \left[u_j^2 \sum_i \frac{dN_j}{dx}\right] dx$$

$$\begin{cases} M\dot{U} + C(U)\,U = 0 \\ M_{ij} = \dfrac{1}{\Delta t}\displaystyle\int_0^1 N_i \sum_j^i N_j \\ C_{ij}(U) = \quad u_j \displaystyle\int_0^1 N_i \sum_i \frac{dN_j}{dx} + \int_0^1 N_i \sum_j \frac{dN}{dx} \end{cases}$$

this system can be solved using Forward / Backward Euler or better, Newton-Raphson

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c) Disscuss how to represent non-linear fluxes:

there are basically three options:

1. use a constant value for each element

$f(u) = f(\bar{u})$ where $\bar{u}$ = average of u in the element.

2. Use the basis functions to obtain the value of u at the location we need

$u^* = \sum_{i=1}^{nsd} N_i(x)\, u_i$ and then use $f(u^*)$

3. Interpolate the value of the flux using flux basis functions

$$f(u) = \sum_{i=1}^{nsd} N_i(x)\, f(u_i)$$

so here the interpolation occurs on the fluxes, not the solution u.

this latter method has 4th order accuracy for linear elements.