Finite Elements in Fluids

Steady convection-diffusion problems

Unsteady convective transport problems

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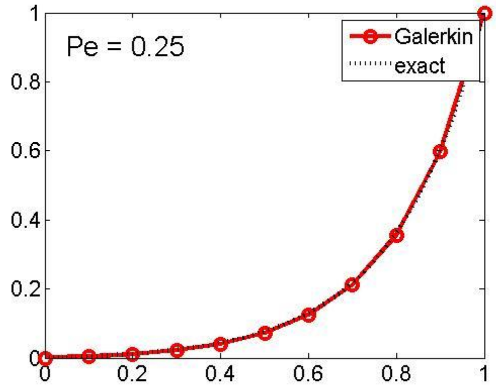
1. INTRODUCTION

The paper is structured as follows: in Section 2 the steady convection-diffusion problem is considered. Galerkin method is tested for different Peclet numbers, then different stabilization techniques with optimal stabilization parameter were tested. 2D steady convection-diffusion-reaction problem was solved The section 3 considers unsteady convective transport problems. For 1D case Crank-Nicolson method for time and a Galerkin formulation for space was implemented and tested for the given initial and analytical solution. Then 2D homogeneous convection equation was considered and different time discretization techniques were discussed.

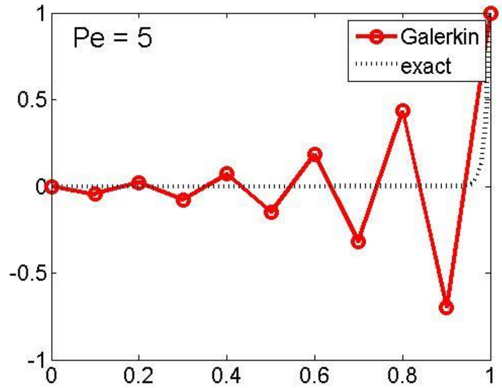
2. STEADY CONVECTION-DIFFUSION PROBLEMS

2.1 1D steady convection-diffusion problems

The first example ($s = 0, u0 = 0, u1 = 1$) was solved using Galerkin method, using different convection velocity a, diffusion coefficient ν and number of linear elements. These different sets of parameters in turn resulted in different Péclet numbers.



(a)



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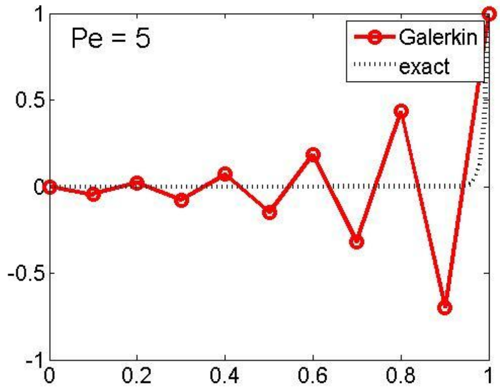

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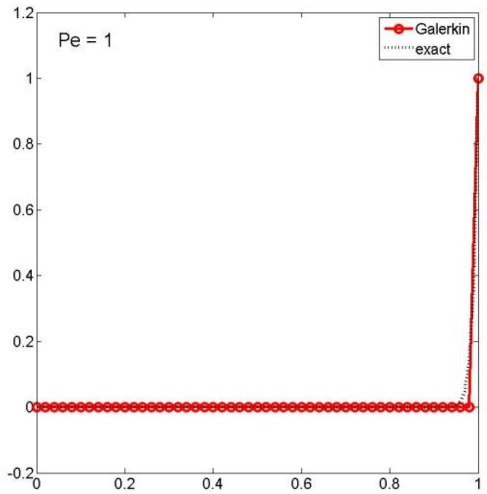


(d)

Fig.1. Comparison of Galerkin method solution for different cases: (a) $a = 1, \nu = 0.2$, 10 linear elements; (b) $a = 20, \nu = 0.2$, 10 linear elements; (c) $a = 1, \nu = 0.01$, 10 linear elements; (d) $a = 1, \nu = 0.01$, 50 linear elements.

As a result, the exact nodal values are obtained only in the case of $Pe = 0{,}25$. At $Pe = 1$ nodal value is not exact, but there are no oscillations, while in other two cases with $Pe = 5$ big oscillations are observed. In these cases of convection-dominated problems with high Péclet number Galerkin loses optimality and spurious effects appear.

In order to eliminate oscillations and obtain exact nodal values at high Péclet numbers, Streamline Upwind (SU), Streamline Upwind Petrov-Galerkin (SUPG), Galerkin Least-Squares (GLS) and Sub-Grid Scale (SGS) stabilization techniques were used.

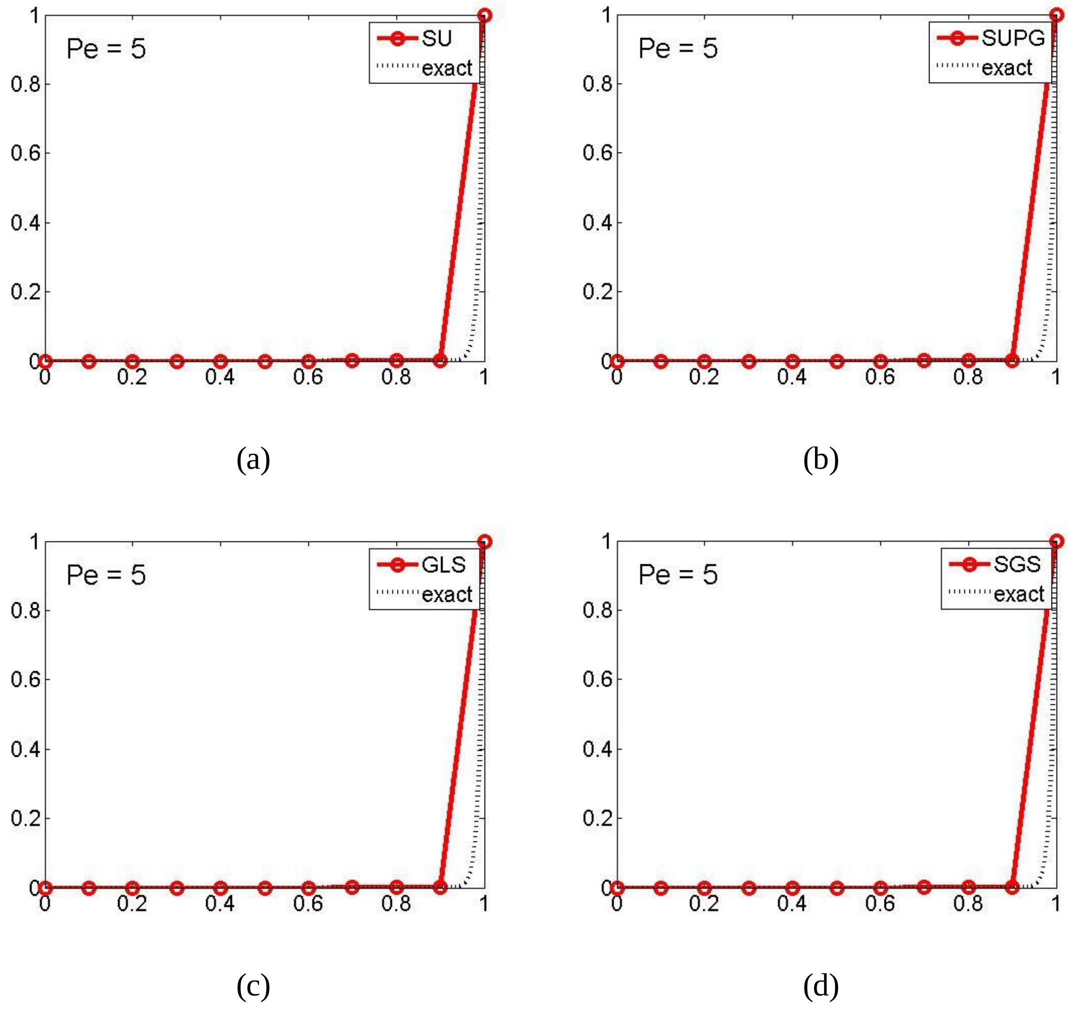
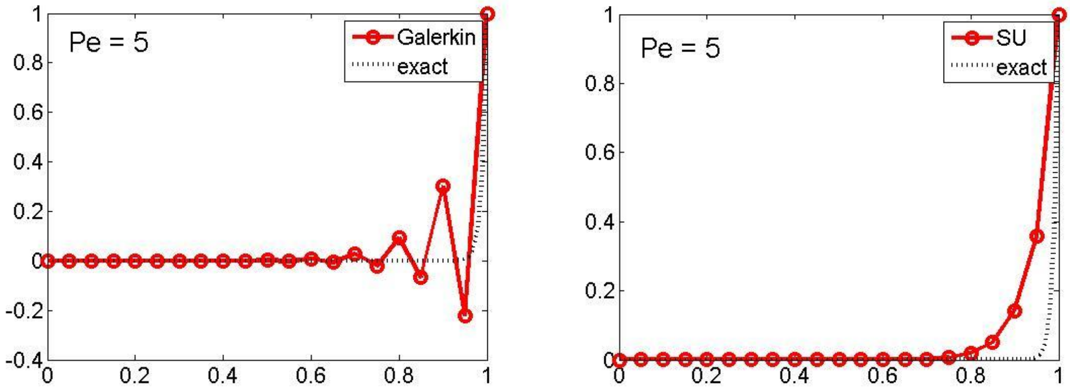


Fig.2. Solution of convection-diffusion problem $a = 1, \nu = 0.01$, 10 linear elements case using (a) Streamline Upwind ; (b) Streamline Upwind Petrov-Galerkin; (c) Galerkin Least-Squares; (d) Sub-Grid Scale stabilization techniques.

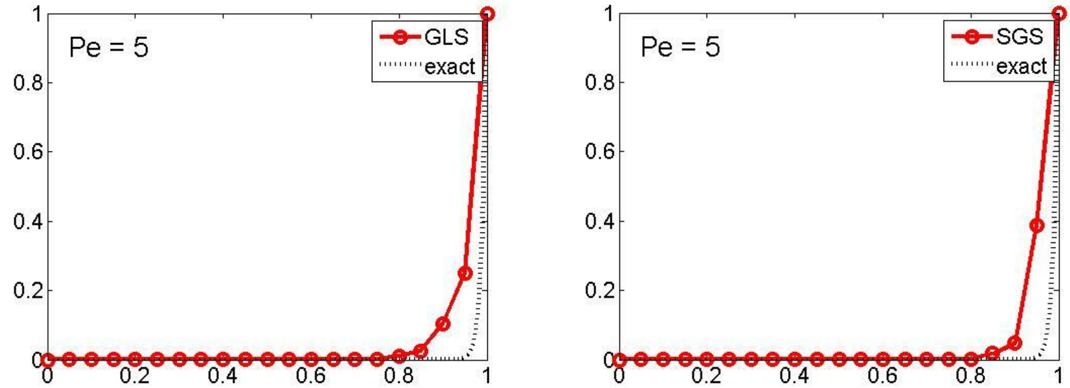
Various high Péclet numbers were tested and the solution was exact for all stabilization techniques.

If the stabilization parameter of Streamline Upwind Petrov-Galerkin is higher than optimal, for instance $\tau = 1$, then convection becomes significantly underestimated, because we have added too much artificial diffusion and made the solution overly diffusive. In case the stabilization parameter is less than optimal, oscillations appear and when $\tau = 0$ we recover the Galerkin solution.

Quadratic elements were implemented, see Fig.3. The results of Galerkin approximation for the mesh of quadratic elements has spurious node-to-node oscillations



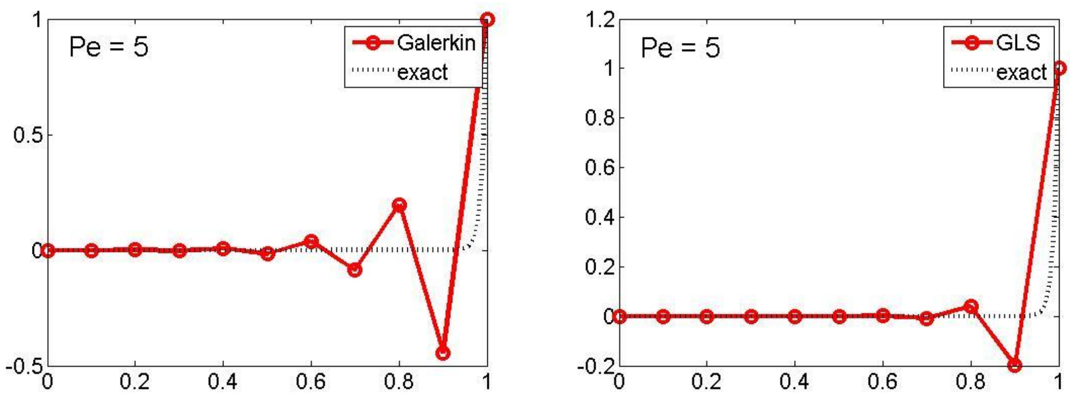
(a) (b)



(c) (d)

Fig.2. Solution of convection-diffusion problem $a = 1, \nu = 0.01, 10$ linear elements case using (a) Streamline Upwind ; (b) Streamline Upwind Petrov-Galerkin; (c) Galerkin Least-Squares; (d) Sub-Grid Scale stabilization techniques.

1.2. 1D steady convection-diffusion-reaction problem



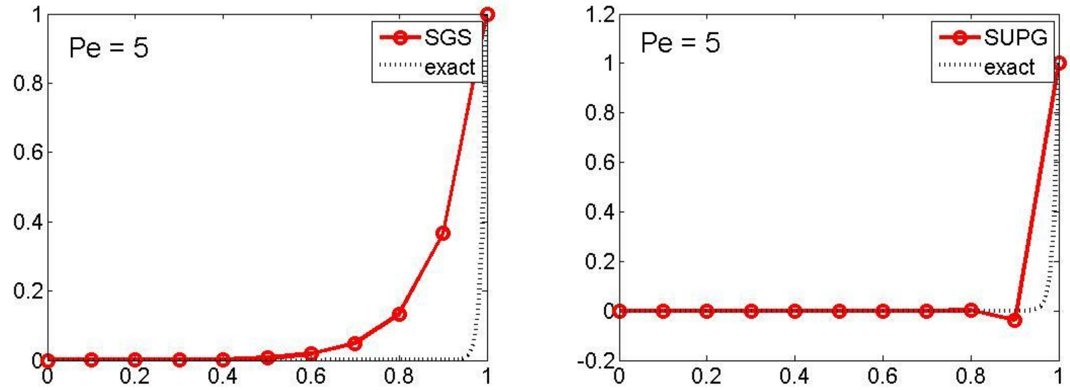


Fig.3. Solution of convection-diffusion-reaction problem $a = 1, \nu = 0.01$, 10 linear elements case using (a) Streamline Upwind ; (b) Streamline Upwind Petrov-Galerkin; (c) Galerkin Least-Squares; (d) Sub-Grid Scale stabilization techniques.

When Galerkin method is refined on the sharp front it performs better, see Fig.4.

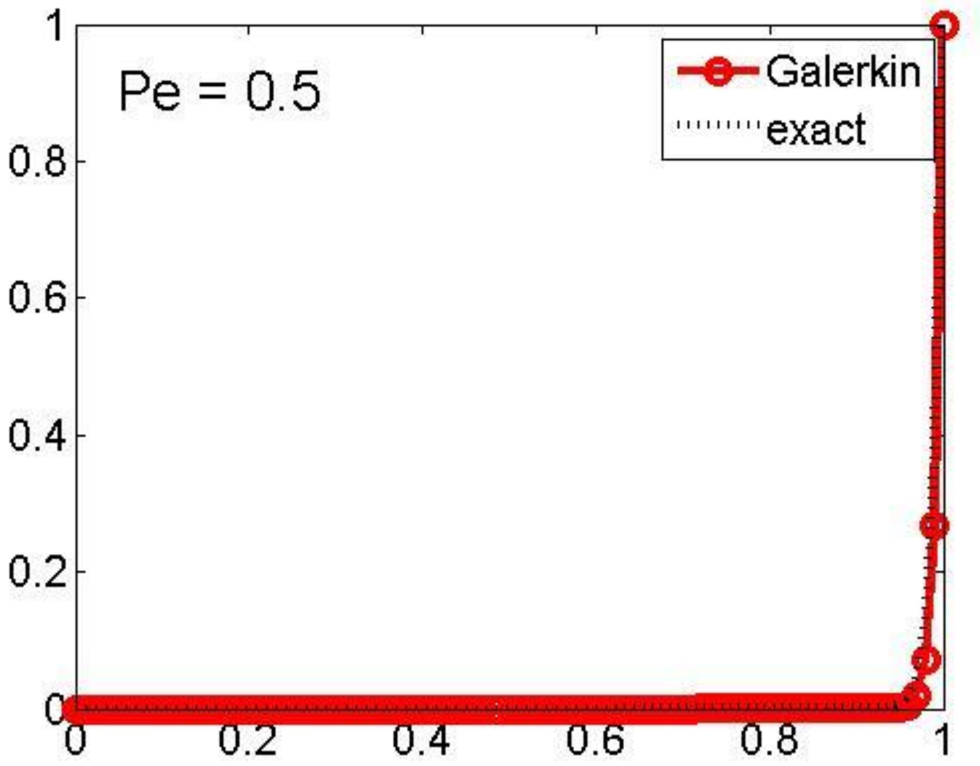
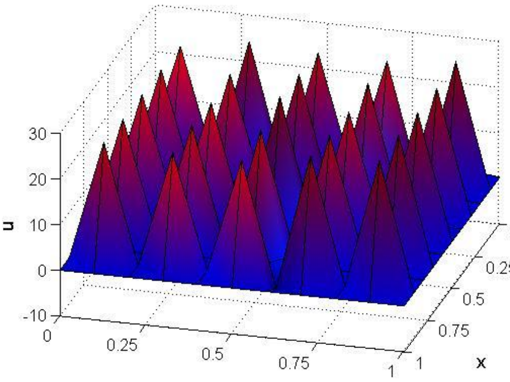


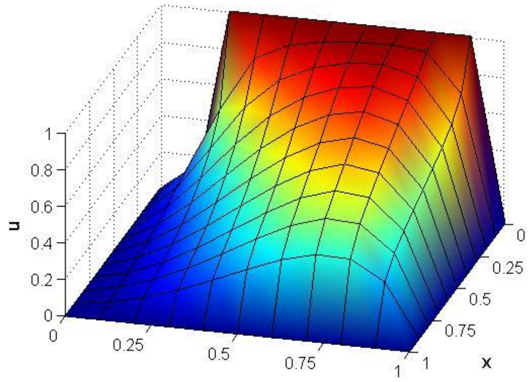
Fig.4. Solution of convection-diffusion-reaction problem $a = 1, \nu = 0.01$, 100 linear elements case using Galerkin method

2.2 1D steady convection-diffusion problems

The MATLAB code was modified in order to impose zero Dirichlet boundary conditions on the outlet side.



(a)



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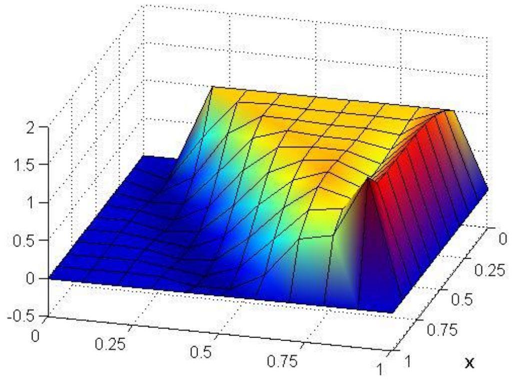
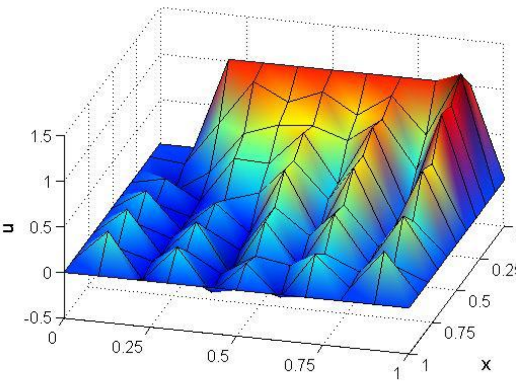


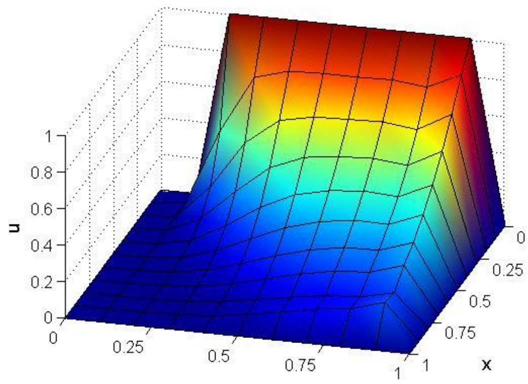
(c)

Fig.5. Galerkin (a), Artificial Diffusion (b) and SUPG (c) solutions of convection-diffusion problem with $||\mathbf{a}|| = 1, \nu = 0.0001, 10$ linear elements and zero Dirichlet boundary conditions on the outlet side.

2D steady convection-diffusion-reaction problem was solved. The reaction term $\sigma = 1$ was considered and the code was modified. The Péclet number in the first case is equal to 250, see Fig.6.



(a)



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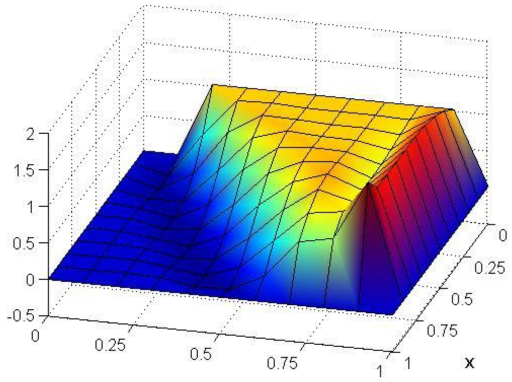


(c)

Fig.6. Galerkin (a), Artificial Diffusion (b) and SUPG (c) solutions of convection-diffusion-reaction problem with $||\mathbf{a}|| = 1/2, \nu = 0.0001$, 10 linear elements and zero Dirichlet boundary conditions on the outlet side.

The Galerkin solution is partially stable, because of low convection velocity $a = 0{,}5$ and therefore low Péclet number. However if the convection velocity is increased $a = 1$, then fully unstable solution is recovered.

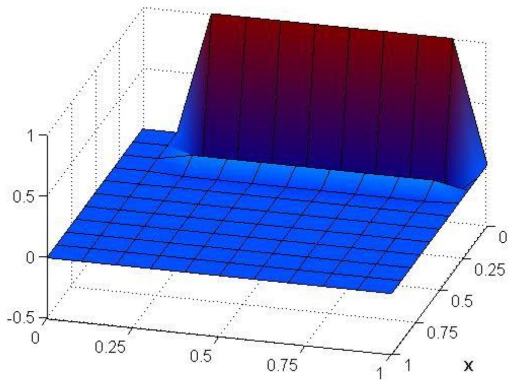
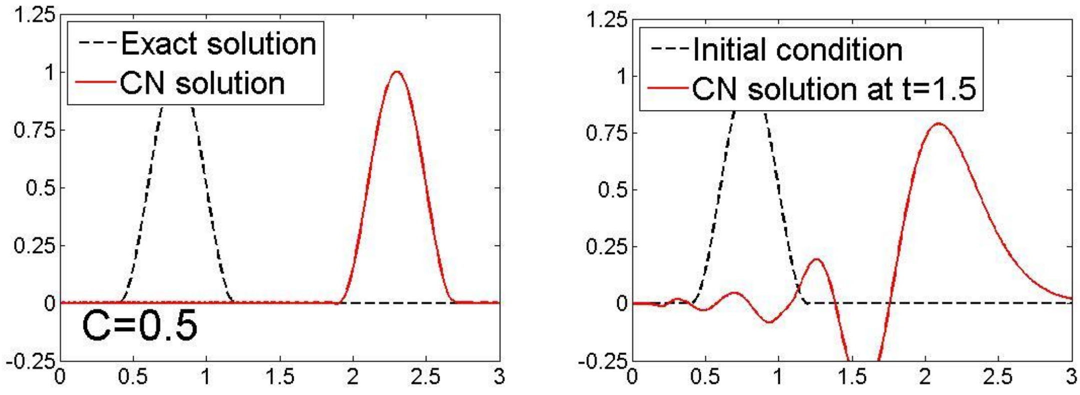


Fig. 7. Reaction dominating problem $||\mathbf{a}|| = 0.001, \nu = 0.0001, \sigma = 1$ 10 linear elements

2. UNSTEADY CONVECTIVE TRANSPORT PROBLEMS

2.1. 1D Unsteady Convective Transport

The code for transient convection equation was modified to solve transient convection-diffusion problem with given initial condition, using Crank-Nicolson method for time and a Galerkin space approximation.



(a) (b)

Fig. 8. Crank-Nicolson solution transient convection equation (a) and transient convection-diffusion equation

The code was used to solve an example with given analytical solution and initial condition:

$$u(x,0) = 5/7 * \exp\left(-\left(\frac{x - x0}{L}\right)^2\right)$$

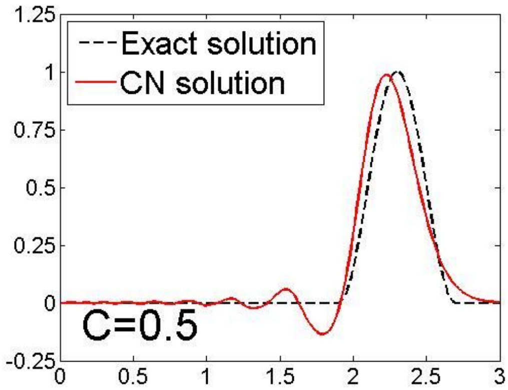


Fig. 9. Crank-Nicolson solution transient convection-diffusion equation (a) and transient convection-diffusion equation

It was observed that for higher Péclet number the oscillations increase.

2.2. 2D HOMOGENEOUS CONVECTION EQUATION

2.2.1. Lax-Wendroff with lumped mass matrix + Galerkin

The pure convection unsteady equation was solved using Lax-Wendroff with lumped mass matrix time discretization with Galerkin space discretization. The initial condition of the following type was considered:

$$u(x,0) = \begin{cases} \frac{1}{4}(1 + cos\pi X1)(1 + cos\pi X2) & if\ X_1^2 + X_2^2 \leq 1, \\ 0 & otherwise \end{cases}$$

The velocity field is $v(x,y) = (-y,x)$. A uniform mesh $20x20$ was used for all simulations and the number of steps was chosen to finish full revolution.

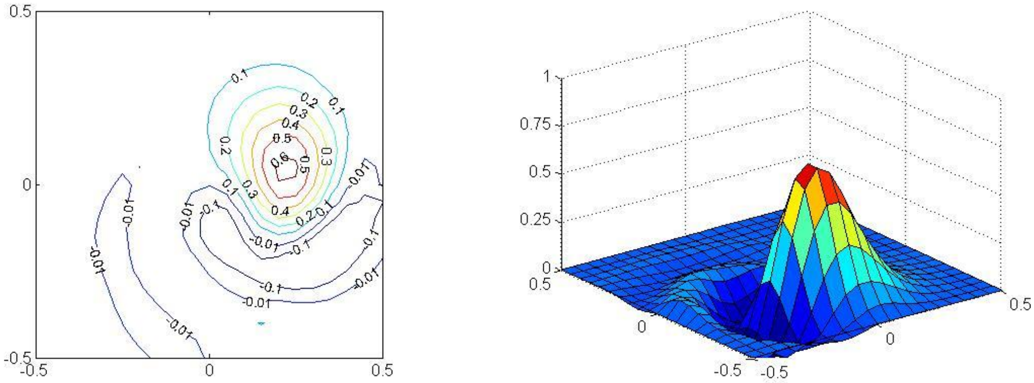


Fig. 10. Isocontours and velocity distribution of transient convection equation using Lax-Wendroff with lumped mass matrix time discretization with Galerkin space discretization

In the case of two other proposed types of the velocity $v(x,y) = (1,0)$ and $v(x,y) = (1,1)/sqrt(2)$ Lax-Wendroff with lumped mass matrix time discretization with Galerkin space discretization also have oscillatory behavior, see Fig. N.

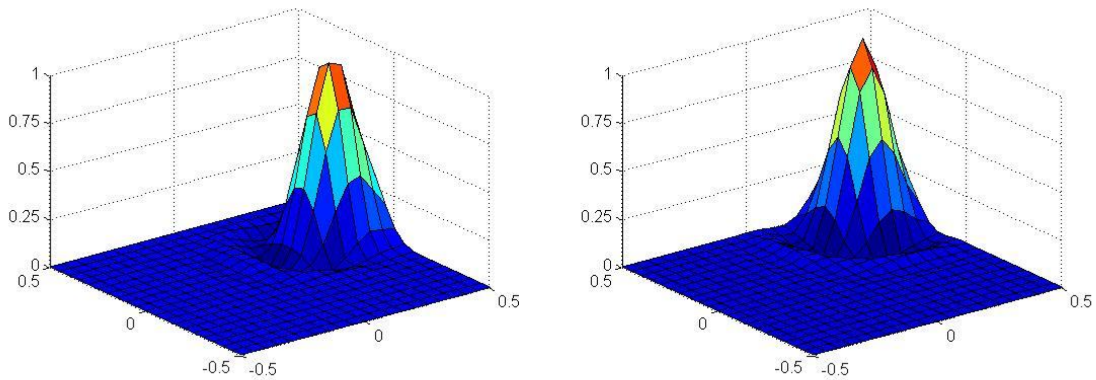


Fig. 11. Solution of transient convection equation using Lax-Wendroff with lumped mass matrix time discretization with Galerkin space discretization

The advantage of Lax-Wendroff with diagonal mass matrix is low computational costs and bigger stability range, but comparing to the Lax-Wendroff with consistent mass matrix it is less accurate, which will be seen in the following subsection.

2.2.2. Lax-Wendroff with consistent mass matrix + Galerkin

To resolve the upper-mentioned shortcomings of oscillations the method was modified to Lax-Wendroff time and Galerkin space discretization. However the computational costs of this technique are higher.

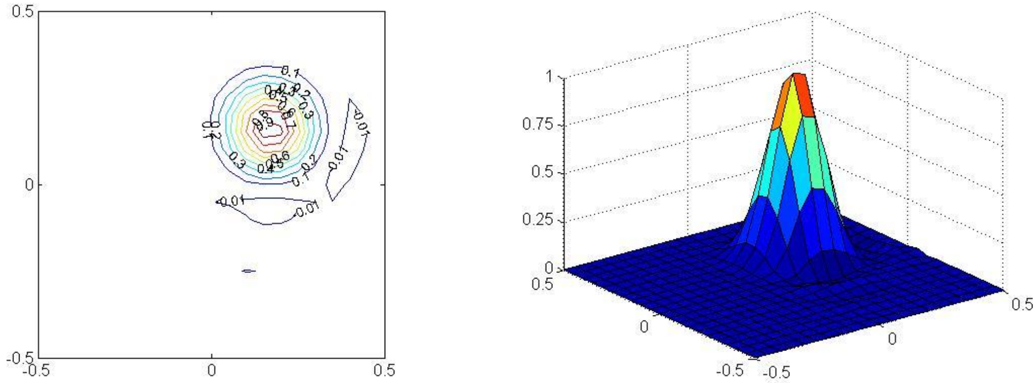


Fig. 12. Isocontours and velocity distribution of transient convection equation using Lax-Wendroff with diagonally dominant mass matrix time and Galerkin space discretization

2.2.3. Crank-Nicolson + Galerkin

A consistent mass matrix was utilized. The code with implemented Crank-Nicolson time and Galerkin space discretization has high accuracy and low oscillatory behavior, maintaining low computational costs at the same time. The

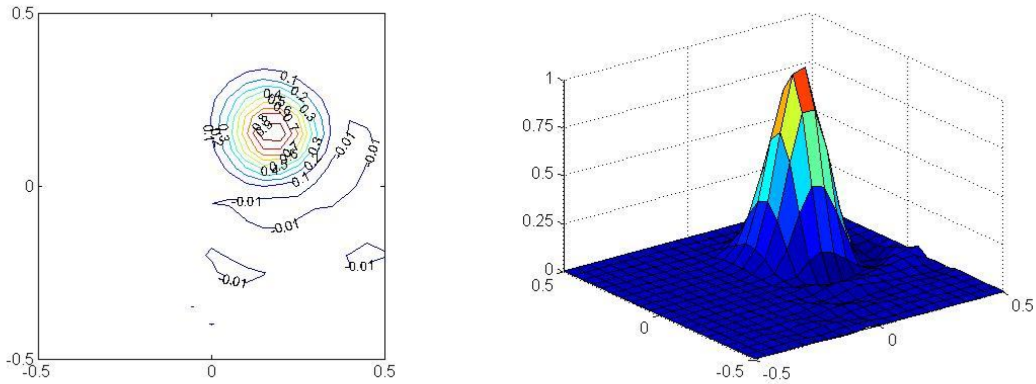


Fig. 13. Isocontours and velocity distribution of transient convection equation using Crank-Nicolson time and Galerkin space discretization

Decrease of the number of the elements leads to higher oscillations for all of the above-mentioned techniques for 2D homogeneous convection equation and even more accurate Lax-Wendroff with consistent mass matrix and Crank-Nicolson techniques has non-accurate solution.