Unsteady Convection-Diffusion Problem

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Ques. Unsteady Pure Convention Problem

A. Compute the Courant number.

B. Solve the problem using the Crank-Nicholson scheme in time and linear finite element for the Galerkin scheme in space. Is the solution accurate?

C. Implement the Crank-Nicholson scheme in time and the least-squares formulation in space. Comment the results.

D. Solve the problem using the second-order Lax-Wendroff method. Can we expect the solution to be accurate? If not, what changes are necessary? Comment the results.

E. Implement the second-order two-step Lax-Wendroff method. Comment the results.

Solution:

Define Problem: **Pure Convection Equation.**

$$u_t + a u_x = 0 \qquad x \in (0,1), t \in (0,0.6]$$

$$u(x,0) = u_0(x) \qquad x \in (0,1)$$

$$u(0,t) = 1 \qquad t \in (0,0.6]$$

$$u_0(x) = 1 \qquad \text{if } x \le 0.2$$

$$u_0(x) = 0 \qquad \text{otherwise}$$

Introduction: To solve defined problem using formulation such as Crank Nicolson and Lax-Wendroff method.

a. Courant Number:

$$C = \| a \| \left(\frac{\Delta t}{\Delta x}\right)$$

To compute courant number, a = 1, Δt = 1.5*10^-2 & Δx = 2*10^-2.

$$C = \| 1 \| \left(\frac{1.5*10^{-2}}{2*10^{-2}}\right) = 0.75$$

b. Crank Nicolson Scheme:

The given problem is solved for Crank-Nicholson scheme in time and linear finite element for the Galerkin scheme in space. It is being observed at courant number C = 0.75, that this method produces many oscillations. So method is unconditionally stable, but not accurate.

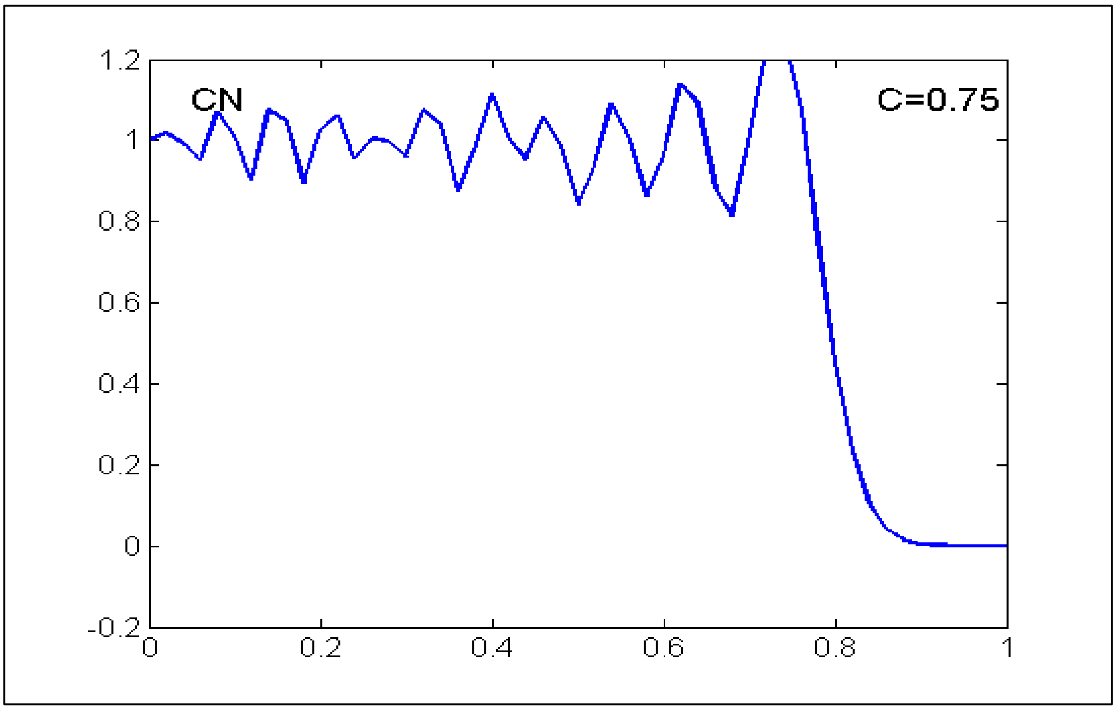

Fig 1: Crank Nicolson Galerkin Formulation

c. Crank Nicolson – Least Square formulation:

The given algorithm of Crank Nicolson Galerkin formulation is modified to Crank Nicolson-Least square formulation in time and space respectively.

```
for i=1:numel
    unos = ones (ngaus,1);
    h = xnode(i+1)-xnode(i);
    xm = (xnode(i)+xnode(i+1))/2;
    weight = wpg*h/2;
    isp = [i i+1];
    % Loop on Gauss points (numerical quadrature)
    for ig=1:ngaus
        N = N_mef(ig,:);
        Nx = Nxi_mef(ig,:)*2/h;
        w_ig = weight(ig);
        x = xm + h/2*xipg(ig); % x-coordinate of the current Gauss point
        % Matrices assembly
        A(isp,isp) = A(isp,isp) + w_ig*(N'*N + dt_2*a*(N'*Nx + Nx'*N + dt_2*a*Nx'*Nx));
        B(isp,isp) = B(isp,isp) -  w_ig*dt*a*(N'*Nx + dt_2*a*Nx'*Nx);
```

Fig 2: Implementation of Crank Nicolson- Least Square formulation

The given problem is solved for Crank-Nicholson scheme in time and least square formulation in space. It is being observed at courant number C = 0.75, that this method tries to reduce oscillations produced by Crank Nicolson Galerkin scheme.

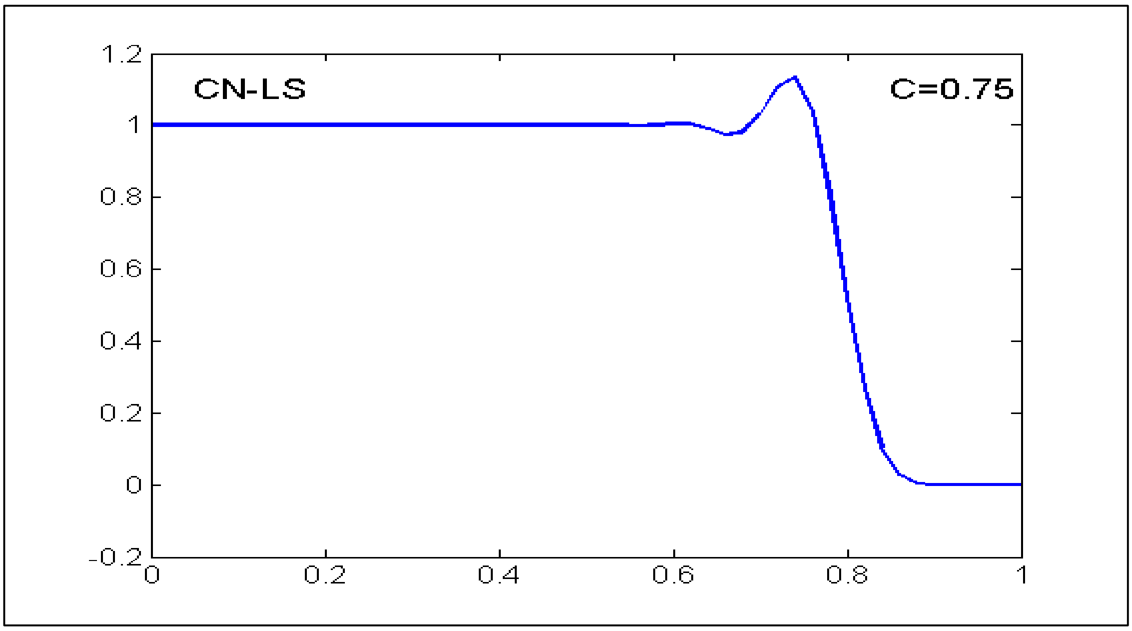

Fig 3: Crank Nicolson Least Square Formulation

d. Lax-Wendroff Method:

It is being observed that Lax-Wendroff method is unstable at courant number C = 0.75. To obtain stability, courant number must to be reduce, as it obtains stability at $C^2 \leq 1/3$.

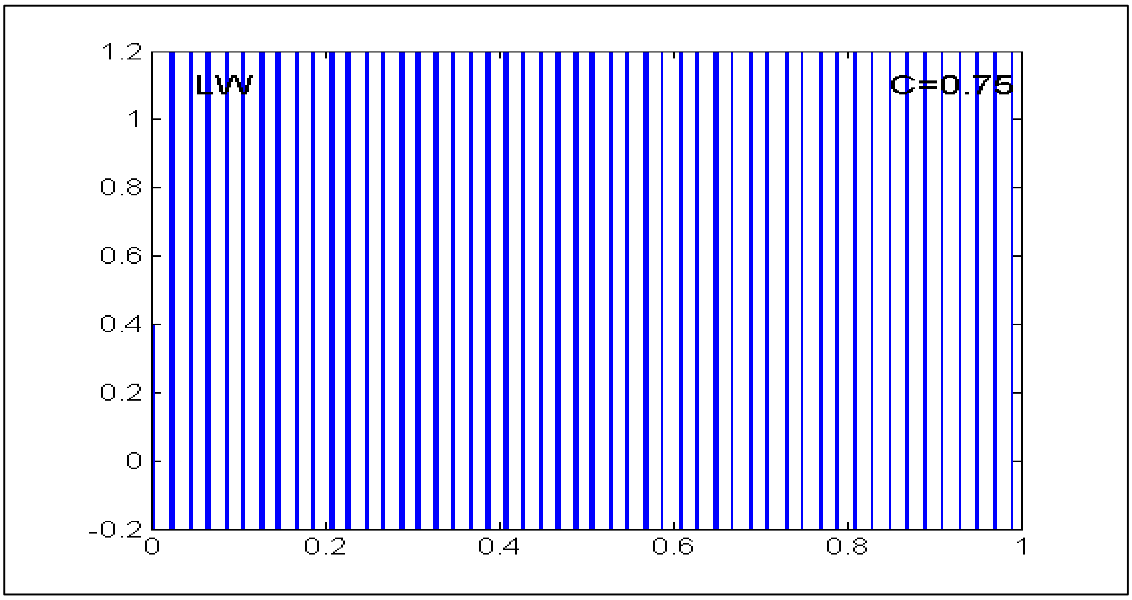

Fig 4: Lax-Wendroff Method

e. Two step Lax-Wendroff Method:

The given algorithm of Lax-Wendroff method is modified to Two-step Lax-Wendroff method.

```
for i=1:numel
    unos = ones (ngaus,1);
    h = xnode(i+1)-xnode(i);
    xm = (xnode(i)+xnode(i+1))/2;
    weight = wpg*h/2;
    isp = [i i+1];
    % Loop on Gauss points (numerical quadrature)
    for ig = 1:ngaus
        N = N_mef(ig,:);
        Nx = Nxi_mef(ig,:)*2/h;
        w_ig = weight(ig);
        x = xm + h/2*xipg(ig); % x-coordinate of the current Gauss point
        % Matrices assembly
        A1(isp,isp) = A1(isp,isp) + w_ig*(N'*N);
        B1(isp,isp) = B1(isp,isp) - w_ig*((dt/2*N'*(a*Nx)));
        f1(isp) = f1(isp) + w_ig*(N')*SourceTerm(x);
        A2(isp,isp) = A2(isp,isp) + w_ig*(N'*N);
        B2(isp,isp) = B2(isp,isp);
        f2(isp) = f2(isp) + w_ig*(N')*SourceTerm(x);
        C2(isp,isp) = C2(isp,isp) - w_ig*(dt*N'*(a*Nx));
```

Fig 5: Implementation of Two-step Lax-Wendroff Method.

It is being observed that Two-step Lax-Wendroff method is unstable at courant number C = 0.75. To obtain stability, modification is required in Discontinuous Galerkin in space and Two-step Lax-Wendroff method in time.

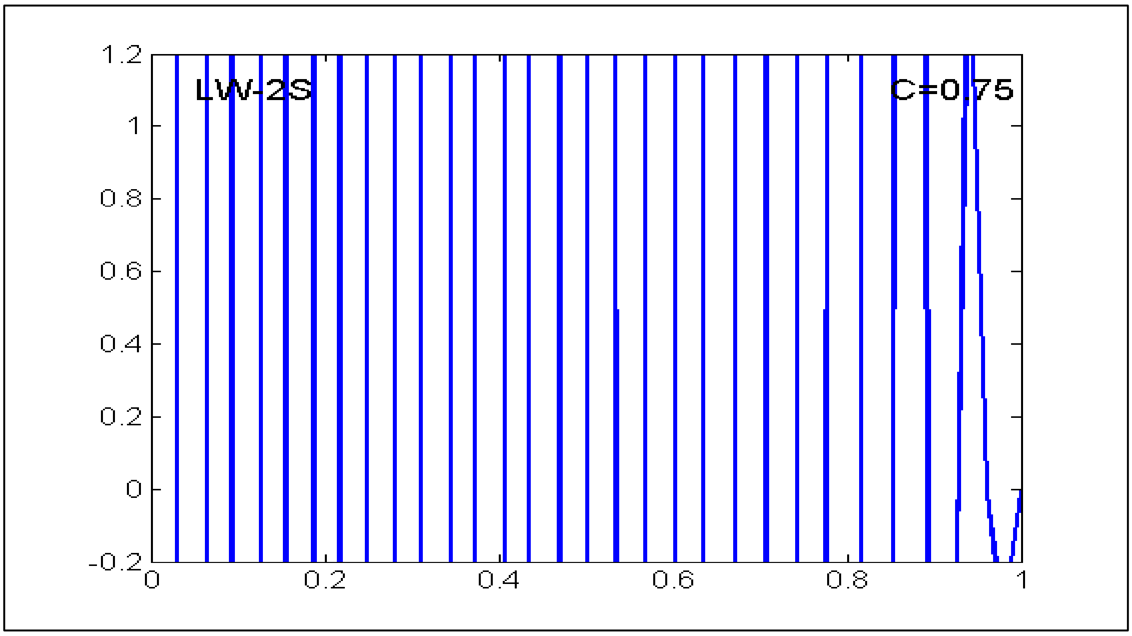

Fig 6: Two-step Lax-Wendroff Method.

-----x-----x-----x-----