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Theoretical and Numerical Study of New Non Local Front Damage Model

by

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A thesis submitted in partial fulfillment for the degree of Erasmus MSc in Computational Mechanics

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Declaration of Authorship

I, DIBAKAR DATTA, declare that this thesis titled, '*Theoretical and Numerical Study of a New Non Local Front Damage Model*' and the work presented in it are my own. I confirm that:

- This work was done wholly or mainly while in candidature for a Erasmus MSc in Computational Mechanics at Ecole Centrale de Nantes(ECN).
- Where any part of this thesis has previously been submitted for a degree or any other qualification at this school or any other institution, this has been clearly stated.
- Where I have consulted the published work of others, this is always clearly attributed.
- Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work.
- I have acknowledged all main sources of help.
- Where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself.

Signed: __________________________________

Date: __________________________________

"I have no special talent.I am only passionately curious."

Prof.Albert Einstein(1879-1955)

Abstract

In this thesis,we have studied the new Non Local Front Damage Model known as Thick Level Set developed by Prof.Moës et al.at ECN,France.Based on this model,we have developed analytical solution for a system with transition damage zone and developed the numerical solution (Finite Difference Solution) of the analytical equation.We have then studied the asymptotic behaviour as the void zone tends to zero(i.e. $d = 1$ only at the center).We have compared the result of Computational (XFEM Code and Finite Difference) and analytical solution.We have developed analytically Force-Displacement relationship for 1D and 2D system.To conclude our study,we have investigated different problems of Mechanics where Thick Level Set Model can be applied.

Acknowledgements

I am highly indebted to my supervisor **Prof.Nicolas Moës** for his astronomical help from day one.His extraordinary insight into the problem help me understand the basic.Because of his amazing guidance, I have learnt how to do research step by step.I must say also about his wonderful personality.
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I am indebted to all my professors and friends around the world.
Of course,thanks to my parents,sisters and all other memebers of my family..

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Abbreviations

Acronym	**W**hat (it) **S**tands **F**or
LFT	**L**egendre **F**enchel **T**ransformation
PE	**P**otential **E**nergy
SED	**S**train **E**energy **D**ensity
FEA	**F**inite **E**lement **A**analysis
ls	**l**evel **s**et
XFEM	e**X**tended **F**inite **E**lement **M**ethod
HDE	**H**eun **D**ifferential **E**quation

Symbols

Symbol	Name	Unit
d	Damage Index	
ϕ	Level Set	m
l_c	Critical length of Level Set	m
$g(s)$	Configurational force	N
ρ	Radius of curvature	m
γ_c	Critical surface energy	$N-m$
φ	Free energy per unit volume	$N-m$
σ	Stress	N/m^2
Y	Local energy release rate	J/m^2
E	Young Modulas	N/m^2
ψ^*	Dissipation potential	$N-m$
Γ_0	Propagating damage front	
Γ_c	Boundary at $\phi = l_c$	
$\overline{Y}_c$	Critical value of configurational force	N
Φ	Airy Stress Function	
P	External outward pressure	N/m^2
U	External outward displacement	m
R_{23}	Reactive force between the undamaged and trasition zone	N

Dedicated to two wonderful persons I have met in my life:

Prof.Nicolas Moës
Dr.Paul-Emile Bernard

Chapter 1

A Brief Overview of New Non Local Front Damage Model

1.1 Introduction

In this chapter,we introduce a new Nonlocal Front Damage Model introduced by Prof.Nicolas MOËS et al at GeM,ECN,France[1].Undamaged Zone,Damaged Zone and Fully Degraded Zone are separated by Level Set.The damage index d is an explicit function of the level set ϕ.Beyond a critical length (l_c), we assume the material to be totally degraded($d = 1$),and it is the transition to fracture. The damage growth is expressed as a level set propagation.The configurational force($g(s)$) driving the damage front is non local in the sense that it averages information over the thickness in the wake of the front.

1.2 A Brief Journey Through Literature

Damage Mechanics is essential to model the gradual loss of stiffness in a small area and the initiation of crack[2].Local damage models suffers from spurious localizations.Several damage models has been proposed in the literature to avoid Spurious Localization[3]:

- non local integral damage model: the damage evolution is governed by a driving force which is non-local i.e. it is the average of the local driving force over some region[4],[5].
- higher order, kinematically based, gradient models through the inclusion of higher-order deformation gradient [6],[7],[8].
- higher order, damage based, gradient models: the gradient of the damage is a variable as well as the damage itself. This leads to a second order operator acting on the damage[9],[10]

More recently,two strategies to avoid spurious localization have also appeared:Phase-field approach in Physics community[11]and variational approach[12].
In this New Nonlocal Front Damage Model,level set is used to track the damage propagation.In report[13],a level set is introduced to separate damaged zone with undamaged zone.It does not introduce a length and will most likely suffer pathological mesh-dependencies known in the computational mechanics community as spurious localization.
Paper [14] uses the same model as [13],with the important difference that a surface energy term is added.The configurational force on the interface must now reach the critical value $Y_c + \gamma_c/\rho$ in which ρ is the radius of curvature of the interface and γ_c a critical surface energy.The model introduces the length scale $l_c = \gamma_c$.The drawback of this model is that the energy to initiate a small hole is infinite since $\rho = 0^+$ at void initiation.
In this model proposed by Prof.N.MOËS et al.,a new regularisation of the local damage model is introduced[1].Within the transition zone, the damage is an explicit function of the level set. This function is a parameter of the model. Beyond a critical distance to the level set front the damage is assumed to be 1. In other words, as the front propagates it unveils in its wake a fully damage zone. We call this zone the crack (although we shall see it is not necessarily of zero thickness).This zone may be of very complex topology, thus dealing easily with branching and merging.In our approach, we thus do not place a crack in a damaged zone as in [15]. The crack appears as a consequence of the damage front motion.

1.3 A New Level Set Based Damage Model(N.Moës et al.)

Free energy per unit volume φ is defined as:

$$\varphi = \varphi(\epsilon, d) \tag{1.1}$$

Stress σ and local energy release rate Y are obtained from φ as:

$$\sigma = \frac{\partial \varphi}{\partial \epsilon}, \qquad Y = -\frac{\partial \varphi}{\partial d} \tag{1.2}$$

For symmetric behavior in traction and compression,we can use the potential

$$\varphi(\epsilon, d) = \frac{1}{2}(1-d)\epsilon : \mathbf{E} : \epsilon \tag{1.3}$$

Which leads to:

$$\sigma = \tfrac{1}{2}(1-d)\epsilon : \mathbf{E} : \epsilon, \qquad Y = \tfrac{1}{2}\epsilon : \mathbf{E} : \epsilon \tag{1.4}$$

The evolution of the damage is given by a dissipation potential $\psi^*(Y)$ convex function of Y:

$$\dot{d} = \frac{\partial \psi^*(Y)}{\partial Y} \tag{1.5}$$

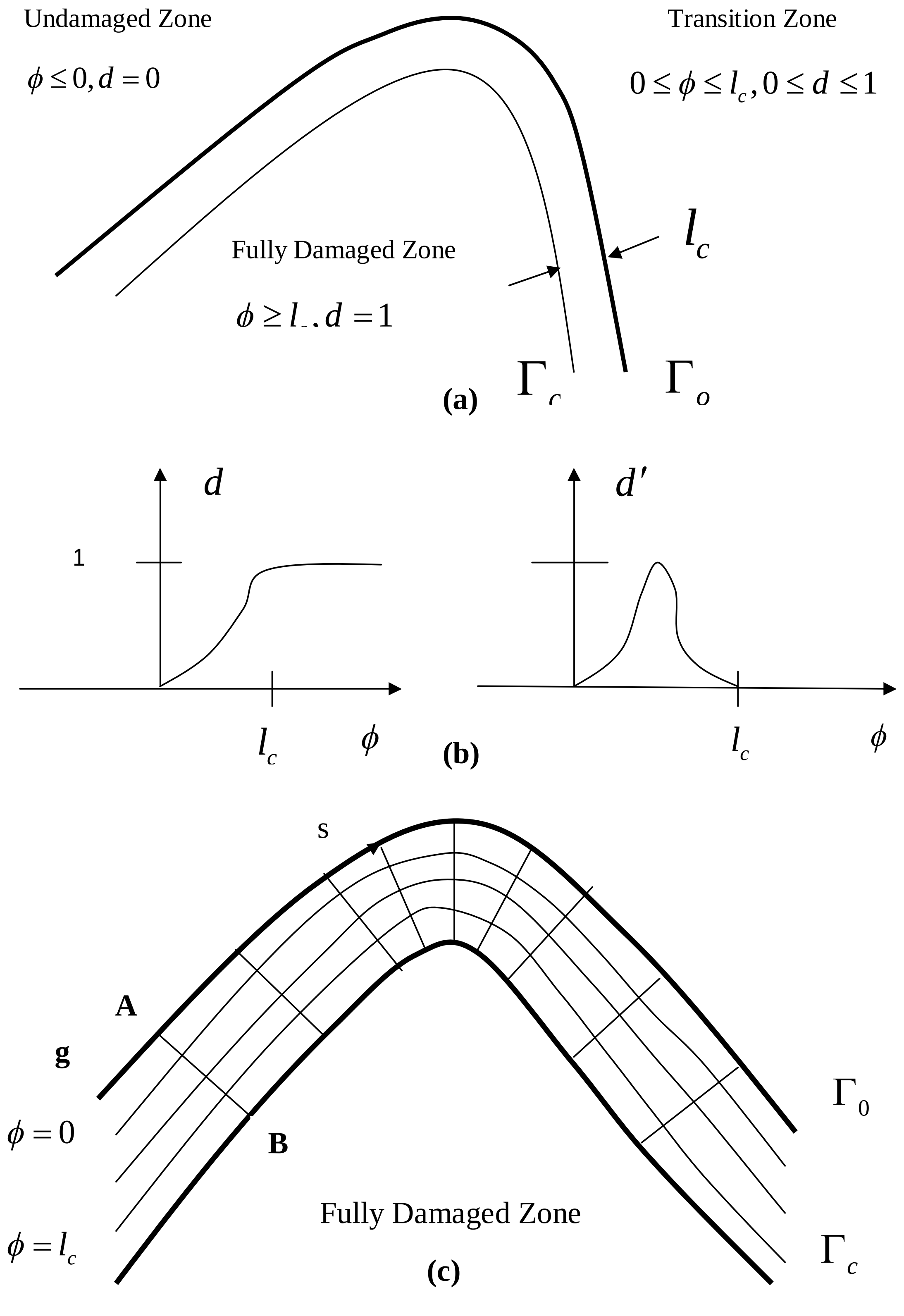

Figure 1.1: (a)Description of the thick level set separating a totally damaged zone from an undamaged zone.(b)Model description of the dependence between the damage variable and the level set variable.The function(left) and its derivative(right). (c)The curvilinear system of coordinate in the band.The front driving force g is defined as the weighted average of the local driving force over the path of steepest descentAB.

Which can be written alternatively as:

$$Y = \frac{\partial \psi(\dot{d})}{\partial \dot{d}} \tag{1.6}$$

The above potentials $\psi(\dot{d})$ and $\partial\psi^*(Y)$ are dual and related by the Legendre-Fenchel transformation:

$$\psi(\dot{d}) = \sup\left(Y\dot{d} - \partial\psi^*(Y)\right)|_Y, \qquad \partial\psi^*(Y) = \sup\left(Y\dot{d} - \partial\psi(\dot{d})\right)|_{\dot{d}} \tag{1.7}$$

Using Equation:1.7,for any couple $(Y, \dot{d})$ we get the following damage evolution law:

$$\psi^*(Y) + \psi(\dot{d}) - Y\dot{d} \geq 0 \tag{1.8}$$

The damage description is depicted in Fig.1.1(a).The level set $\phi = 0$ separates the domain Ω into an undamaged and damaged zone.In the damaged zone,the damage variable d is an explicit function of the level set.The type of relation governing damage index is depicted in Fig.1.1(b).The damage increases progressively as the level set value rises which can be expressed as:

$$d(\phi) = 0, \quad \phi \leq 0 \tag{1.9}$$

$$d^{'}(\phi) \geq 0, \quad 0 \leq \phi \leq l_c \tag{1.10}$$

$$d(\phi) = 1, \quad \phi \geq l_c \tag{1.11}$$

At a distance l_c from the damage front,the material is assumed to be completely damaged.Considering a body Ω, the part not fully damaged(Ω_c) can be expressed as:

$$\Omega_c = \{x \in \Omega : \phi(x) \leq l_c\} \tag{1.12}$$

Following the procedure described by Prof.N Moës et el.[1],we can show that the configurational force per unit length on the front $(g(s))$ can be given by:

$$g(s) = \int_0^l Y(\phi, s) d^{'}(\phi) \left(1 - \frac{\phi}{\rho(s)}\right) d\phi \tag{1.13}$$

In Equation:1.13,l is either l_c or some smaller value.
The configurational force(1.13)is an average of the local damage driving force Y weighted by $d^{'}(\phi)$ and the evolution of the front curvature along the thickness.
An illustraion of this averaging is depicted in Fig.1.1(c) along the path AB.

$$g(A) = \int_{AB} Y d^{'}(\phi) \left(1 - \frac{\phi}{\rho(A)}\right) d\phi \tag{1.14}$$

The critical value of the configurational force ($\overline{Y}_c(s)$) is given by:

$$\overline{Y}_c(s) = Y_c(s) \int_0^l Y(\phi, s) d^{'}(\phi) \left(1 - \frac{\phi}{\rho(s)}\right) d\phi \tag{1.15}$$

Damage front propagates when $g(s)$ reaches its critical value ($\overline{Y}_c(s)$).

1.4 X-FEM Code

In this present study,we have compared our semi-analytical(Finite Difference solution of the analytical equation) and analytical solution (Heun and Kummer Solution) with the solution obtained from X-FEM code. The code has been developed by Prof.Nicolas Moës[1] Group at Ecole Centrale de Nantes(ECN),Nantes,France.

The X-FEM(eXtended Finite Element Method)[16][17] is a numerical method for modelling strong

(displacement) and weak(strain) discontinuities within the standard Finite Element framework.The conventional Finite Element Method is problemetic in many practical cases e.g. modeling of moving discontinuities because we need to update the mesh in order to track the geometry of discontiuity.The X-FEM approach allows us to model moving geometry without remeshing.

The X-FEM code developed at ECN is periodically updated to introduce new features of this topic.It is used to model crack and damage propagation in materials.For the present study we have used this code for our model in order to validate the accuracy of our work.

1.5 Organization of the Thesis

In summary,in this chapter,we have introduced a new way to model damage growth in solids developed by N.Moës Group at ECN,France.The model is based on so called *Thick Level Set*.The damage variable is tied to the distance to damage front(iso-zero of the level set).Beyond a critical distance the material is considered completely damaged (but may still sustain compression if it is prescribed in the model).The model thus introduces a length scale and it regularizes local damage models.The work in remaining thesis is primarily based on this model.

The rest of the thesis has been structured in the following fashion:

Chapter 2: In this chapter entitled *Numerical and Analytical Solution for Circular Axisymmetric Plate with Transition Damage Zone*,we will study the governing equations for the fields of interest for the undamaged and damaged zone.We will derive the compatibility and equilibrium equation for the damaged zone and perform the Finite Difference Solution

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of the analytical equation.We will then investiagte the analytical solution by Hypergeometric,Kummer and Heun Function.We will conclude the Chapter giving numerical result with comparison with the result form the X-FEM code and discussion about the result.

Chapter 3: In this chapter entitled *Asymptotic Analysis of Displacement , Stress and Strain Field for Void Zone Tending Zero($d = 1$ at the center)*,we will study the behaviour of the displacement,stress and strain field for the case when material at the center point only is fully degraded i.e.$d = 1$ is only at the center.We will investigate that Equidimensional approach will yield erroneous result.Like the previous Chapter,We will conclude the Chapter giving numerical result with comparison with the result form the X-FEM code and discussion about the result.

Chapter 4: In this chapter entitled *Analytical Force-Displacment Relationship Based on Growth law in Thick Level Set Approcah*,we will study the force and displacement as a function of the propagating damage length.For the one dimensional case,we will study the influence of different damage law and length of critical level set.We will extend our study for the two dimensional model considered in Chapter 2 and develope analytically using the Heun Solution , the equation of Force and Displacement as a function of propagating damage length.Numerical exmaples will demonstrate the variation of stiffness w.r.t. the damage propgation,force required to initiate damage propagation for a given damage length and the corresponding displacement and the Force-Displacement relation with damage propagation.At the end, we will note an algorithm to develope Froce-Displacement relation by the numerical solution of the analytical solution.

Chapter 5: In this chapter entitled *Recommendation for Future Work*,we will take a short tour in the world of Mechanics where the newly developed 'Thick Level Set' can be applied.We will study four different topics: Stability Analysis of Crack Propagation,Equilibrium Shape of Propagating Damage Zone,Application in Geophysics and for the Localization problems in Plasticity.

Appendix A: We will derive the Compatibility Equation for the damaed zone.

Appendix B: We will derive the Equilibrium Equation for the undamaged zone[2].

Appendix C: We will take a quick look at the Heun Function and Heun Differential Equation as we will use it extensively for the developement of our analytical solution.

Appendix D: We will derive the expression of the reactive force at the junction between the damaged and the undamaged zone and the coeffecients associated with the analytical expression given by Heun Equation.

Appendix E: We will derive analytically the Force-Displacement relationship for the one-dimensional case for different damage laws.

[2]Note that for the undamaged zone this is the well-known Lamé Equation

Chapter 2

Numerical and Analytical Solution for Circular Axisymmetric Plate with Transition Damage Zone

2.1 Introduction

In Chapter(1),we have given an overview of the new Non Local Damage Model called Thick Level Set Model.In this chapter, we have given an analytical solution for the stress,strain and displacement field for a circular plate with transition damage zone. In the transition zone i.e. the damaged zone, the Youngs Modulus is not constant, but is given as $E(1-d)$,where d is the damage index. Hence it has to be taken into account in the compatibility equation. For the undamaged zone, the analytical solution is obtained using the conventional Airy Stress approach as described in any textbook on Solid Mechanics e.g.[18]. For the transition zone, there are two possibilities either using the compatibility equation using Airys Stress function or formulating an equation for displacement using the stress-strain relationship and the equilibrium equation.

Ref.to fig(2.1(a)),we find that the system has three distinct zone[1]:

$$\begin{cases} \text{d=1} & : \text{Zone1}\{\text{Fully Degraded Zone(Radius } r = r_1)\} \\ \text{d=}\frac{r_2 - r}{l_c} & : \text{Zone2}\{\text{Damaged Zone(Inner Radius } r = r_1 \text{ and Outer Radius } r = r_2)\} \\ \text{d=0} & : \text{Zone3}\{\text{Undamaged Zone(Inner Radius } r = r_2 \text{ and Outer Radius } r = r_3)\} \end{cases} \tag{2.1}$$

The level set value(ϕ) is as described in Chapter(1).The structure is subjected to either external displacement or pressue.

[1] Note that r_1 does not exist for the case of no void i.e.max$(d) < 1$

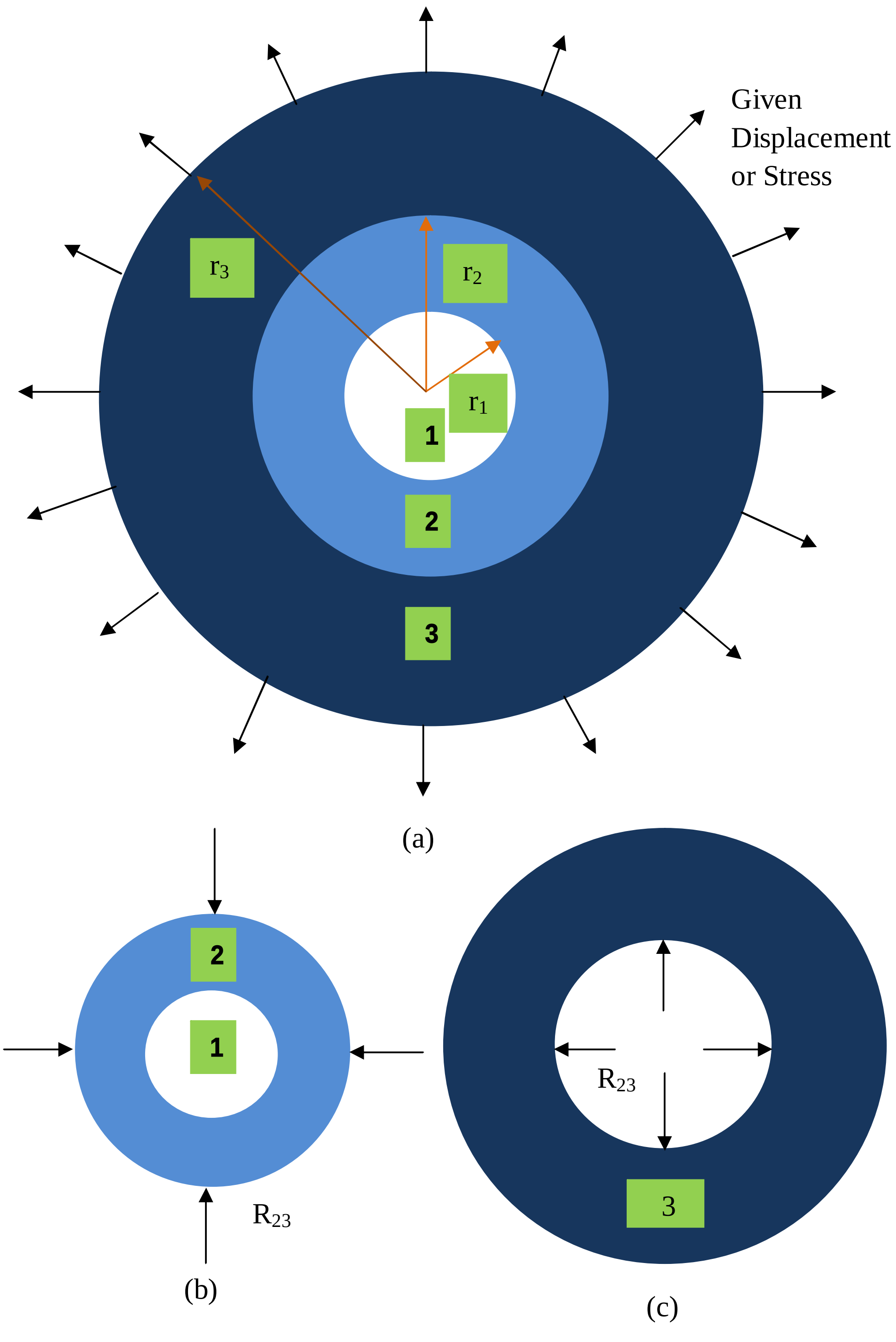


Figure 2.1: (a)Circular Plate with void having a transition damage zone and subjected to axisymmetric loading or dispalcement.(b)Inner damaged zone subjected to reaction R_{23} at the contact.(c)Outer undamaged zone

2.2 Governing Equations for Undamaged Zone

Since the problem is axisymmetric,the Compatibility Equation is given by[18]:

$$\nabla^4\Phi = \Big(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\Big)\Big(\frac{\partial^2\Phi}{\partial r^2} + \frac{1}{r}\frac{\partial\Phi}{\partial r}\Big) \tag{2.2}$$

Here Φ is the Airy Stress Function.[18] The solution of the above equation is given by:

$$\nabla\Phi = A\log(r) + Br^2\log(r) + Cr^2 + D \tag{2.3}$$

Boundary Condition:

Ref. to fig(2.1),we write the boundary condition for the undamaged section as:

$$\begin{cases} \text{At } r = r_3 & : \quad \sigma_{rr} = P \text{ or } u_r = U \\ \text{At } r = r_2 & : \quad \sigma_r r = -R_{23} \end{cases} \tag{2.4}$$

Where:

P	:	External Outward Pressue
U	:	External Outward Displacement
R_{23}	:	Reactive force at the junction between Zone2 and Zone3

The reactive force R_{23} can be obtained by equating the displacement for Zone:2 and Zone:3 at the junction i.e. at $r = r_2$.The additional constants can be determined by comparing the expression of displacement from strain in radial and transversal direction.

Displacement , Stress and Strain Field for the Undamaged Zone[2]

1. **Case 1:**Boundary Condition at $r = r_3 : \sigma_{rr} = -P$

   Displacement:

$$u = \frac{1}{E}\Bigg\{-\frac{r_2^2 r_3^2(P - R_{23})}{r_3^2 - r_2^2}\frac{(1+\nu)}{r} + \frac{R_{23}r_2^2 - Pr_3^2}{r_3^2 - r_2^2}(1+\nu)r\Bigg\} \tag{2.5}$$

   Radial Stress:

$$\sigma_{rr} = \frac{r_2^2 r_3^2(P - R_{23})}{r_3^2 - r_2^2}\frac{1}{r^2} + \frac{R_{23}r_2^2 - Pr_3^2}{r_3^2 - r_2^2} \tag{2.6}$$

   Hoop Stress[3]:

$$\sigma_{\theta\theta} = -\frac{r_2^2 r_3^2(P - R_{23})}{r_3^2 - r_2^2}\frac{1}{r^2} + \frac{R_{23}r_2^2 - Pr_3^2}{r_3^2 - r_2^2} \tag{2.7}$$

[2]The procedure of derivation for this kind of problem can be found in any text book on Solid Mechanics.Hence,only the final expressions are written.

[3]Note that sum of Radial and Hoop Stress (and the strain)is constant.

Radial Strain:

$$\varepsilon_{rr} = \frac{1}{E}\left\{\frac{r_2^2 r_3^2 (P - R_{23})}{r_3^2 - r_2^2}\frac{(1+\nu)}{r^2} + \frac{R_{23} r_2^2 - P r_3^2}{r_3^2 - r_2^2}(1+\nu)\right\} \tag{2.8}$$

Hoop Strain:

$$\varepsilon_{\theta\theta} = \frac{1}{E}\left\{-\frac{r_2^2 r_3^2 (P - R_{23})}{r_3^2 - r_2^2}\frac{(1+\nu)}{r^2} + \frac{R_{23} r_2^2 - P r_3^2}{r_3^2 - r_2^2}(1+\nu)\right\} \tag{2.9}$$

2. **Case 2:**Boundary Condition at $r = r_3 : u_{rr} = -U$

   Displacement:

$$u = \frac{(R_{23} r_2^2 (1+\nu) + UEr_3) r (1-\nu) + \frac{r_2^2 r_3 (EU - R_{23} r_3 (1-\nu))(1+\nu)}{r}}{E(r_3^2 (1-\nu) + r_2^2 (1+\nu))} \tag{2.10}$$

   Radial Stress:

$$\sigma_{rr} = \frac{1}{1-\nu^2}\left(\frac{-\left(R_{23} r_2^2 (1+\nu) + UEr_3\right)(1-\nu^2) + r_2^2 r_3 (EU - R_{23} r_3 (1-\nu^2))\left(1 + \frac{\nu}{r^2}\right)}{(r_3^2 (1-\nu) + r_2^2 (1+\nu))}\right) \tag{2.11}$$

   Hoop Stress:

$$\sigma_{\theta\theta} = \frac{1}{1-\nu^2}\left(\frac{-\left(R_{23} r_2^2 (1+\nu) + UEr_3\right)(1-\nu^2) - r_2^2 r_3 (EU - R_{23} r_3 (1-\nu^2))\left(\frac{1-\nu}{r^2}\right)}{(r_3^2 (1-\nu) + r_2^2 (1+\nu))}\right) \tag{2.12}$$

   Radial Strain:

$$\varepsilon_{rr} = \frac{(R_{23} r_2^2 (1+\nu) + UEr_3)(1-\nu) - \frac{r_2^2 r_3 (EU - R_{23} r_3 (1-\nu))(1+\nu)}{r^2}}{E(r_3^2 (1-\nu) + r_2^2 (1+\nu))} \tag{2.13}$$

   Hoop Strain:

$$\varepsilon_{\theta\theta} = \frac{(R_{23} r_2^2 (1+\nu) + UEr_3)(1-\nu) + \frac{r_2^2 r_3 (EU - R_{23} r_3 (1-\nu))(1+\nu)}{r^2}}{E(r_3^2 (1-\nu) + r_2^2 (1+\nu))} \tag{2.14}$$

2.3 Governing Equations for the Damaged Zone

2.3.1 Compatibility Equation for the Damaged Zone

2.3.1.1 General Case(Damage Index $d = f(r, \theta)$)

The compatibility euation in polar coordinate is given by the following equation[18]:

$$\frac{\partial^2 \varepsilon_\theta}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 \varepsilon_r}{\partial \theta^2} + \frac{2}{r}\frac{\partial \varepsilon_\theta}{\partial r} - \frac{1}{r}\frac{\partial \varepsilon_r}{\partial r} - \frac{1}{r}\frac{\partial^2 \gamma_{r\theta}}{\partial r \partial \theta} - \frac{1}{r}\frac{\partial \gamma_{r\theta}}{\partial \theta} = 0 \tag{2.15}$$

Compatibility Equation for the damaged zone when the damage index $d = f(r, \theta)$,can be derived as[4]:

$$g_1 \frac{\partial^4 \Phi}{\partial r^4} + g_2 \frac{\partial^3 \Phi}{\partial r^3} + g_3 \frac{\partial^2 \Phi}{\partial r^2} + g_4 \frac{\partial \Phi}{\partial r} + g_5 \frac{\partial^3 \Phi}{\partial \theta \partial r^2} + g_6 \frac{\partial^4 \Phi}{\partial r^2 \partial \theta^2} + g_7 \frac{\partial^2 \Phi}{\partial r \partial \theta} + g_8 \frac{\partial^3 \Phi}{\partial r \partial \theta^2} + g_9 \frac{\partial \Phi}{\partial \theta} + g_{10} \frac{\partial^2 \Phi}{\partial \theta^2} + g_{11} \frac{\partial^3 \Phi}{\partial \theta^3} + g_{12} \frac{\partial^4 \Phi}{\partial \theta^4} = 0 \tag{2.16}$$

Where:

$$\begin{aligned}
&g_1 = f_1 && g_2 = f_2 + \tfrac{1}{r} f_4 && g_3 = \tfrac{1}{r} f_3 - \tfrac{3}{r^2} f_4 + f_5 \\
&g_4 = -\tfrac{1}{r^2} f_3 + \tfrac{2}{r^3} f_4 + \tfrac{1}{r} f_6 && g_5 = f_{10} - \tfrac{1}{r} f_{13} && g_6 = \tfrac{1}{r^2} f_4 + f_9 - \tfrac{1}{r} f_{11} \\
&g_7 = \tfrac{1}{r} f_8 + \tfrac{2}{r^2} f_{13} - \tfrac{1}{r} f_{14} && g_8 = -\tfrac{1}{r^2} f_3 - \tfrac{4}{r^3} f_4 + \tfrac{1}{r} f_7 + \tfrac{2}{r^2} f_{11} - \tfrac{1}{r} f_{12} && g_9 = -\tfrac{2}{r^3} f_{13} + \tfrac{1}{r^2} f_{14} \\
&g_{10} = -\tfrac{2}{r^3} f_3 + \tfrac{6}{r^4} f_4 + \tfrac{1}{r^2} f_6 - \tfrac{2}{r^3} f_{11} + \tfrac{1}{r^2} f_{12} && g_{11} = \tfrac{1}{r^2} f_8 && g_{12} = \tfrac{1}{r^2} f_7
\end{aligned}$$

$$\begin{aligned}
&f_1 = \tfrac{1}{E} && f_2 = -\tfrac{2}{E^2}\tfrac{\partial E}{\partial r} + \tfrac{2+\nu}{Er} \\
&f_3 = \tfrac{2\nu}{E^2}\tfrac{\partial E}{\partial r} - \tfrac{1+2\nu}{Er} && f_4 = -\tfrac{\nu}{E} \\
&f_5 = -\tfrac{1}{E^2}\tfrac{\partial^2 E}{\partial^2 r} + \tfrac{2}{E^3}\left(\tfrac{\partial E}{\partial r}\right)^2 + \tfrac{\nu}{(Er)^2}\tfrac{\partial^2 E}{\partial \theta^2} - \tfrac{2\nu}{E^3 r^2}\left(\tfrac{\partial E}{\partial \theta}\right)^2 - \tfrac{2+\nu}{E^2 r}\tfrac{\partial E}{\partial r} && \\
&f_6 = \tfrac{\nu}{E^2}\tfrac{\partial^2 E}{\partial^2 r} - \tfrac{2\nu}{E^3}\left(\tfrac{\partial E}{\partial r}\right)^2 + \tfrac{2}{E^3 r^2}\left(\tfrac{\partial E}{\partial \theta}\right)^2 - \tfrac{1}{(Er)^2}\tfrac{\partial^2 E}{\partial \theta^2} + \tfrac{1+2\nu}{E^2 r}\tfrac{\partial E}{\partial r} && \\
&f_7 = \tfrac{1}{Er^2} && f_8 = -\tfrac{2}{(Er)^2}\tfrac{\partial E}{\partial \theta} \\
&f_9 = -\tfrac{\nu}{Er^2} && f_{10} = \tfrac{2\nu}{(Er)^2}\tfrac{\partial E}{\partial \theta} \\
&f_{11} = -\tfrac{2(1+\nu)}{Er} && f_{12} = \tfrac{2(1+\nu)}{E^2 r}\tfrac{\partial E}{\partial r} - \tfrac{2(1+\nu)}{Er^2} \\
&f_{13} = \tfrac{2(1+\nu)}{E^2 r}\tfrac{\partial E}{\partial \theta} && f_{14} = \tfrac{2(1+\nu)}{E^2 r}\tfrac{\partial^2 E}{\partial r \partial \theta} - \tfrac{4(1+\nu)}{E^3 r}\tfrac{\partial E}{\partial r}\tfrac{\partial E}{\partial \theta} + \tfrac{2(1+\nu)}{(Er)^2}\tfrac{\partial E}{\partial \theta}
\end{aligned}$$

[4] Derivation has been given in Appendix:A

2.3.1.2 Axisymmetric Case(Damage Index $d = f(r)$ only)

For the axisymmetric case,using the Airy Stress Function approach,for the case when damage index $d = f(r)$ only,the compatibility equation can be simplified to:

$$\frac{\partial^4\Phi}{\partial r^4} + \alpha_3(r)\frac{\partial^3\Phi}{\partial r^3} + \alpha_2(r)\frac{\partial^2\Phi}{\partial r^2} + \alpha_1(r)\frac{\partial\Phi}{\partial r} = 0 \tag{2.17}$$

Where:

$$\left| \begin{array}{lcl} \alpha_3(r) & = & \frac{2}{r} + \left[-\frac{2}{E}\frac{\partial E}{\partial r}\right] \\ \alpha_2(r) & = & -\frac{1}{r^2} + \left[-\frac{1}{E}\frac{\partial^2 E}{\partial^2 r} + \frac{2}{E^2}\left(\frac{\partial E}{\partial r}\right)^2 - \frac{(2-\nu)}{Er}\frac{\partial E}{\partial r} + \frac{\nu}{r^2}\right] \\ \alpha_1(r) & = & \frac{1}{r^3} + \left[\frac{\nu}{Er}\frac{\partial^2 E}{\partial^2 r} - \frac{2\nu}{E^2 r}\left(\frac{\partial E}{\partial r}\right)^2 + \frac{1}{Er^2}\frac{\partial E}{\partial r}\right] \end{array} \right.$$

Here the terms in [ ] in the expresion of α are the extra terms due to damage[5].

One approach to determine the fields of interest for the damaged zone is to solve the above equation for Φ.Considering the linear damage index as in Equation:2.1, Equation:2.17 becomes:

$$\frac{\partial^4\Phi}{\partial r^4} + \left(\frac{2}{r} + \frac{2}{r_2 - r}\right)\frac{\partial^3\Phi}{\partial r^3} + \left(-\frac{1}{r^2} + \frac{2}{(r_2-r)^2} + \frac{(2-\nu)}{(r_2-r)r} + \frac{\nu}{r^2}\right)\frac{\partial^2\Phi}{\partial r^2} + \left(\frac{1}{r^3} - \frac{2\nu}{(r_2-r)^2 r} - \frac{1}{(r_2-r)r^2}\right)\frac{\partial\Phi}{\partial r} = 0 \tag{2.18}$$

The above equation can be solved numerically.But the equation has no analytical solution.Since we are also interested to have an anayltical expressions for the damaged zone besides the semi analytical equation,we consider the equilibrium equation as described in next section.

2.3.2 Equilibrium Equation for the Damaged Zone

Equlibrium Equation for the undamaged zone is the well-known Lamé Equation:

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{1}{r^2}u = 0 \tag{2.19}$$

The Equilibrium Equation for the damaged zone can be derived as[6]:

$$\frac{\partial^2 u}{\partial r^2} + \beta_1(r)\frac{\partial u}{\partial r} + \beta_0(r)u = 0 \tag{2.20}$$

Where:

$$\left| \begin{array}{lcl} \beta_1(r) & = & \frac{1}{r} + \left[\frac{1}{E}\frac{\partial E}{\partial r}\right] \\ \beta_0(r) & = & -\frac{1}{r^2} + \left[\frac{\nu}{Er}\frac{\partial E}{\partial r}\right] \end{array} \right.$$

[5]Note that when $E \neq d(f(r))$,we get back the Equation: 2.2 i.e.Compatibility Equation for undamaged zone

[6]Derivation has been given in Appendix:B

Here the terms in [ ] in the expresion of β are the extra terms due to damage[7].

2.4 Numerical Solution of Analytical Equation

In Section.2.3.2 , we have studied the Equilibrium Equation for the damaged and undamaged zone given by Equation.2.19 and Equation.3.2.2.In this section , we will develope the solution of this sytem of equations numerically using Finite Difference method.

2.4.1 Discretization of the Governing Equation

We rewrite the Equation.2.19,i.e.Equilibrium Equation for the undamaged zone:

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{1}{r^2}u = 0 \tag{2.21}$$

In Fig.2.2(a),we have mentioned the cell for the discretization method we have followed.Considering Finite Difference Approach,we can write:

$$\frac{d^2u}{dr^2}|_i = \frac{u_{i+1} - 2u_i + u_{i-1}}{(\Delta r)^2} \tag{2.22}$$

$$\frac{du}{dr}|_i = \frac{u_{i+1} - u_{i-1}}{2\Delta r} \tag{2.23}$$

Using Equation.2.22 and Equation.2.23,Equation.2.21 can be reshaped as:

$$\underbrace{\left[\frac{1}{(\Delta r)^2} + \frac{1}{r_i}\frac{1}{2\Delta r}\right]}_{\beta_1} u_{i+1} + \underbrace{\left[-\frac{2}{(\Delta r)^2} - \frac{1}{r_i^2}\right]}_{\beta_2} u_i + \underbrace{\left[\frac{1}{(\Delta r)^2} - \frac{1}{r_i}\frac{1}{2\Delta r}\right]}_{\beta_3} u_{i-1} = 0 \tag{2.24}$$

We rewrite the Equilibrium Equation for the damaged zone given by Equation.3.2.2 as:

$$\frac{d^2u}{dr^2} + \left[\frac{1}{r} + \frac{1}{E}\frac{dE}{dr}\right]\frac{du}{dr} + \left[-\frac{1}{r^2} + \frac{k}{Er}\frac{dE}{\partial r}\right]u = 0 \tag{2.25}$$

Here $k = \nu$ for Plane Stress and $k = \frac{\nu}{1-\nu}$ is for the Plane Strain. Considering damage index d as given by Equation.2.1,we can write: $\frac{1}{E}\frac{dE}{dr} = \frac{1}{l_c+r-r_2}$. Using this result and the

[7]Note that when $E \neq d(f(r))$,we get back the Equation: 2.19 i.e.The Lamé Equation

discretization as given by Equation.2.22 and Equation.2.23,we can reshape Equation.2.25 as:

$$\underbrace{\left[\frac{1}{(\Delta r)^2}+\frac{1}{2\Delta r}\left(\frac{1}{r_i}+\frac{1}{r_i-(r_2-l_c)}\right)\right]}_{\alpha_1}u_{i+1}+\underbrace{\left[-\frac{2}{(\Delta r)^2}+\left(-\frac{1}{r_i^2}+\frac{k}{r_i}\frac{1}{r_i-(r_2-l_c)}\right)\right]}_{\alpha_2}u_i+ \\ \underbrace{\left[\frac{1}{(\Delta r)^2}-\frac{1}{2\Delta r}\left(\frac{1}{r_i}+\frac{1}{r_i-(r_2-l_c)}\right)\right]}_{\alpha_3}u_{i-1}=0 \tag{2.26}$$

2.4.2 Boundary Condition

1. Boundary Condition at location $i = 0$ (Fig.2.2(a)) i.e. at the beginning of the damage zone.

   - **IF**

     $\max(d) < 1$ i.e.for the case when there is no void or the fully degraded zone about to initiate,we can write $u|_{r=0} = 0$.Displacement at the center of the system is zero due to symmetry of the system.

   - **ELSE**

     We have the case of void zone. In this case we can write the radial stress at the free surface is zero i.e.$\sigma_{rr}|_{r=0} = 0$

2. Boundary Condition at location $i = N$ (Fig.2.2(a)) i.e. at the end of the undamaged zone.

   In this case,we impose a known value of displacement U_N at $r = r_3$.

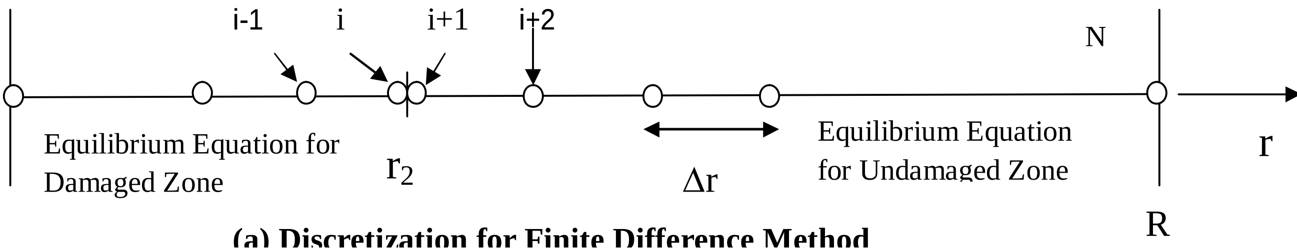
How to choose U_N to satisfy a given condition $\sigma_{rr}|_{r=r_3} = P$?
We will compare our solution with the solution obtained from X-FEM Code(1.4) where we impose external outward pressure to propagate the damage front.Hence we have a given condition $\sigma_{rr}|_{r=r_3} = P$.We need to determine the displacement that will satisfy the given condition.
We write the equlibrium equation of the system as:

$$\nabla\cdot\sigma = 0 \qquad \Rightarrow \qquad \int_\Omega \nabla\cdot\sigma d\Omega = 0 \tag{2.27}$$

Using Divergence Theorem, we can write the above equation as:

$$\int_\Gamma (\sigma\cdot n)\cdot u d\Gamma = \int_\Omega \sigma : \varepsilon d\Omega \tag{2.28}$$

Using Equation.2.28,we can determine the pressure at $r = r_3$ for a given $u(r = r_3) = U_N$ as:



(a) Discretization for Finite Difference Method

$$\begin{pmatrix}
\alpha_2^1 & \alpha_1^1 & & & & & & & & & & \\
\alpha_3^2 & \alpha_2^2 & \alpha_1^2 & & & & & & & & & \\
 & \alpha_3^3 & \alpha_2^3 & \alpha_1^3 & & & & & & & & \\
 & & .... & .... & ... & & & & & & & \\
 & & & & 1 & -1 & & & & & & \\
 & & & 1/2 & 0 & -1 & 1/2 & & & & & \\
 & & & & & \beta_3^{i+2} & \beta_2^{i+2} & \beta_1^{i+2} & & & & \\
 & & & & & & \beta_3^{i+3} & \beta_2^{i+3} & \beta_1^{i+3} & & & \\
 & & & & & & & ... & ... & ... & & \\
 & & & & & & & & ... & ... & ... & \\
 & & & & & & & & & \beta_3^{N-2} & \beta_2^{N-2} & \beta_1^{N-2} \\
 & & & & & & & & & & \beta_3^{N-1} & \beta_2^{N-1}
\end{pmatrix}
\begin{Bmatrix} U_1 \\ U_2 \\ .. \\ .. \\ U_i \\ U_{i+1} \\ U_{i+2} \\ U_{i+3} \\ .. \\ \\ U_{N-2} \\ U_{N-1} \end{Bmatrix}
=
\begin{Bmatrix} -\alpha_3^1 U_0 \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ -\beta_1^{N-1} U_N \end{Bmatrix}$$

(b) Au=b system for the case max(d)<1, U_0=0

$$\begin{pmatrix}
\alpha_0 & 1/\Delta r & & & & & & & & & & & \\
\alpha_3^1 & \alpha_2^1 & \alpha_1^1 & & & & & & & & & & \\
 & \alpha_3^2 & \alpha_2^2 & \alpha_1^2 & & & & & & & & & \\
 & & ... & ... & ... & & & & & & & & \\
 & & & ... & ... & ... & & & & & & & \\
 & & & & & 1 & -1 & & & & & & \\
 & & & & 1/2 & 0 & -1 & 1/2 & & & & & \\
 & & & & & & \beta_3^{i+2} & \beta_2^{i+2} & \beta_1^{i+2} & & & & \\
 & & & & & & & \beta_3^{i+3} & \beta_2^{i+3} & \beta_1^{i+3} & & & \\
 & & & & & & & & ... & ... & ... & & \\
 & & & & & & & & & ... & ... & ... & \\
 & & & & & & & & & & \beta_3^{N-2} & \beta_2^{N-2} & \beta_1^{N-2} \\
 & & & & & & & & & & & \beta_3^{N-1} & \beta_2^{N-1}
\end{pmatrix}
\begin{Bmatrix} U_0 \\ U_1 \\ U_2 \\ .. \\ .. \\ U_i \\ U_{i+1} \\ U_{i+2} \\ .. \\ .. \\ \\ U_{N-2} \\ U_{N-1} \end{Bmatrix}
=
\begin{Bmatrix} 0 \\ 0 \\ .. \\ .. \\ \\ \\ \\ \\ \\ \\ 0 \\ 0 \\ -\beta_1^{N-1} U_N \end{Bmatrix}$$

(c) Au=b system for the void case σ_r (r=0)=0

FIGURE 2.2: (a)Discretization for the implementation of Finite Difference Method.(b)Formulation of the Matrix $Au = b$ for the case $\max(d) < 1$ i.e.$U_0 = 0$.(c)Formulation of the Matrix $Au = b$ for the case of void i.e.$\sigma_{rr}|_{r=0} = 0$.

$$P|_{r=r_3} = f(U_N) = \frac{1}{r_3 U_N} \int_0^{r_3} (\sigma_{rr}\varepsilon_{rr} + \sigma_{\theta\theta}\varepsilon_{\theta\theta})\, rdr \tag{2.29}$$

Hence the algorithm to determine U_N to satisfy a given condition $\sigma_{rr}|_{r=r_3} = P$ can be summarized as:

- Select arbitrary value of U_N.
- Compute $P|_{r=r_3}$ using Equation.2.29.
- Compute error using Equation:

$$\text{error} = \frac{F(U_N) - P|_{r=r_3}}{P|_{r=r_3}} \tag{2.30}$$

- Check the tolerence.
  IF Error within tolerence,**END**
  ELSE Update U_N and **GOTO** Step 2.

2.4.3 Formulation of $Au = b$ System

We will write α_k^i and β_k^i as the $k^{\text{th}}(k = 1, 2, 3)$ coeffecient of α(Equation.2.26) and β (Equation.2.24) at the i^{th} node (Fig.2.2(a)).

1. Condition of No Void ($\max(d) < 1, u(r = 0) = 0$).
   The system of equation $Au = b$ is given by Fig.2.2(b).In this case we know $u(r = 0) = U_0$.Hence our unknow vector is $u_j, j = 1 \dots N - 1$.
   To apply Boundary Condition at i and i+1 location (Fig.2.2(a)),we write $A(i+1, i+1) = 1$,$A(i + 1, i + 1) = -1$,$A(i + 2, i) = \frac{1}{2}$,$A(i + 2, i + 2) = -1$,$A(i + 2, i + 3) = \frac{1}{2}$

2. Condition of Void ($\sigma_{rr}|_{r=0} = 0$)
   The system of equation $Au = b$ is given by Fig.2.2(c).In this case we do not know $u(r = 0) = U_0$.Hence our unknow vector is $u_j, j = 0 \dots N - 1$.
   We know that $\sigma_{rr}|_{r=0} = 0$.It can be writen in discretized form as:

$$\frac{1}{\Delta r}U_1 + \underbrace{\left[-\frac{1}{\Delta r} + \frac{k}{r_0}\right]}_{\alpha_0} U_0 = 0 \tag{2.31}$$

   We can apply the boundary conditions at the interface as before.

2.5 Analytical Solution Using Hypergeometric Function

Considering the linear variation of damage $d = 1 - \frac{r - r_2 + l_c}{l_c}$,Equation:3.2.2 can be reshaped as:

$$r^2 \frac{\partial^2 u}{\partial r^2} + \left(r + \frac{r^2}{(r - r_2 + l_c)} \right) \frac{\partial u}{\partial r} + \left(-1 + \frac{kr}{(r - r_2 + l_c)} \right) u = 0 \tag{2.32}$$

Where $k = \nu$ for Plane Stress and $k = \frac{\nu}{1-\nu}$ for Plane Strain. The Equation.2.32 can be solved using Hypergeometric Function as:

$$\begin{aligned} u = C_1 \text{hypergeom} \left(\left[\frac{3}{2} - \frac{1}{2}\sqrt{5-4k}, -\frac{1}{2} - \frac{1}{2}\sqrt{5-4k} \right], \left[1 - \sqrt{5-4k} \right], \frac{r_2 - l_c}{r} \right) r^{-\frac{1}{2} + \frac{1}{2}\sqrt{5-4k}} \\ + C_2 \text{hypergeom} \left(\left[\frac{3}{2} + \frac{1}{2}\sqrt{5-4k}, -\frac{1}{2} + \frac{1}{2}\sqrt{5-4k} \right], \left[1 + \sqrt{5-4k} \right], \frac{r_2 - l_c}{r} \right) r^{-\frac{1}{2} - \frac{1}{2}\sqrt{5-4k}} \end{aligned} \tag{2.33}$$

2.5.1 Case of No Void($u_{rr}|_{r=0} = 0$)

For this case,we know that displacement at the center of the system is zero i.e.$u_{rr}|_{r=0} = 0$ due to the symmetry of the problem.Considering Equation.2.33,we find that in order to satisfy this condition,we should have $C_2 = 0$ since $r^{-\frac{1}{2}-\frac{1}{2}\sqrt{5-4k}} \to \infty$ at $r = 0$.Even if we go ahead with that approach,we do not get the desired solution as the Hypergeometric Function $\to \infty$ as $r \to 0$.The singularity influences a large neighbourhod of $r = 0$. Hence analytical solution using Hypergeometric Function for this case is not possible.

2.5.2 Case of Void($\sigma_{rr}|_{r=0} = P$)

Singularity of Solution: We have solved the problem using Equation.2.33 for the case of void.We write $r_1 = r_2 - l_c$ as the radius of the void zone.Because of the presence of $r - r_1$ term in the denomenator,the solution is singular at $r = r_1$.The hypergeomtric function is not stable at a large neighbourhood of $r = r_1$.Because the function starts moving toward infinite value from a large distance of $r = r_1$.As a result it influences the whole solution as depicted in Fig.2.3.

We have considered the following numerical values considered are: $l_c = 0.03$m, $r_2 = 0.05$m, $r_3 = 0.1$m, $E = 37.7 \times 10^9 \frac{N}{m^2}$, $\nu = 0.20$, $P = 1 \times 10^7 \frac{N}{m^2}$.Here $k = \nu$, i.e. we are in Plane Stress not in Plane Strain.

From Fig.2.3,we observe **erroneous result because of singularity.**

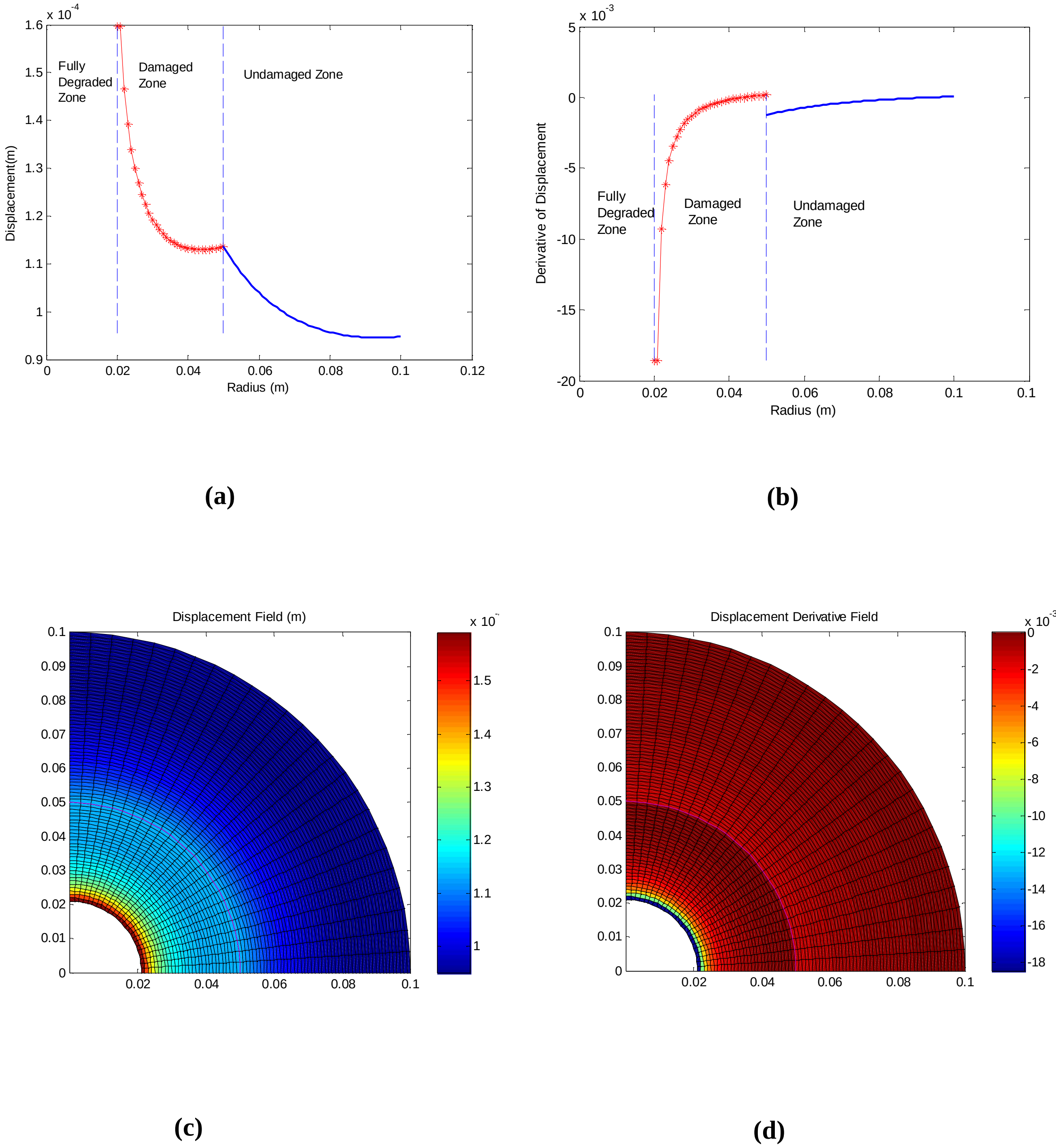


FIGURE 2.3: **ERRONEOUS RESULT** using Hypergeometric function because of singularity.

2.6 Analytical Solution Using Kummer Function

We replace $r_2 - l_c = r_1$ in Equation.2.32.In order to kill the singularity,we approximate the term $r - r_1$in Equation.2.32 as:

$$\frac{1}{r-r_1} = \frac{1}{r}\left(1-\frac{r_1}{r}\right)^{-1} = 1 + \left(\frac{r_1}{r}\right) + \left(\frac{r_1}{r}\right)^2 + \cdots \tag{2.34}$$

If we consider only the linear term ,we will get the following **Kummer Differential Equation** as:

$$r^2\frac{\partial^2 u}{\partial r^2} + \left(r + r\left(1+\frac{r_1}{r}\right)\right)\frac{\partial u}{\partial r} + \left(-1 + k\left(1+\frac{r_1}{r}\right)\right)u = 0 \tag{2.35}$$

The solution of the above equation can be given as:

$$\begin{aligned} u = {} & C_1\text{KummerM}\left(\frac{1}{2}+\frac{1}{2}\sqrt{5-4k}-k, 1+\sqrt{5-4k}, \frac{r_1}{r}\right) r^{-\frac{1}{2}-\frac{1}{2}\sqrt{5-4k}} \\ & + C_2\text{KummerU}\left(\frac{1}{2}+\frac{1}{2}\sqrt{5-4k}-k, 1+\sqrt{5-4k}, \frac{r_1}{r}\right) r^{-\frac{1}{2}-\frac{1}{2}\sqrt{5-4k}} \end{aligned} \tag{2.36}$$

2.7 Analytical Solution Using Heun Function

If we consider only the square term as in Equation.2.34 ,we will get the following **Heun Differential Equation** as:

$$r^2\frac{\partial^2 u}{\partial r^2} + \left(r + r\left(1+\frac{r_1}{r}+\left(\frac{r_1}{r}\right)^2\right)\right)\frac{\partial u}{\partial r} + \left(-1 + k\left(1+\frac{r_1}{r}+\left(\frac{r_1}{r}\right)^2\right)\right)u = 0 \tag{2.37}$$

The solution of the above equation can be given as:

$$\begin{aligned} u = {} & C_1\text{HeunB}\left(\sqrt{5-4k}, \sqrt{2}, 1+2k, -2k\sqrt{2}, \frac{1}{2}\frac{\sqrt{2}r_1}{r}\right) r^{-\frac{1}{2}-\frac{1}{2}\sqrt{5-4k}} \\ & + C_2\text{HeunB}\left(-\sqrt{5-4k}, \sqrt{2}, 1+2k, -2k\sqrt{2}, \frac{1}{2}\frac{\sqrt{2}r_1}{r}\right) r^{-\frac{1}{2}+\frac{1}{2}\sqrt{5-4k}} \end{aligned} \tag{2.38}$$

2.7.1 Case of No Void ($u_{rr}|_{r=0} = 0$)

For this case,we take $C_1 = 0$ to kill the singularity at $r = 0$ due to the term $r^{-\frac{1}{2}-\frac{1}{2}\sqrt{5-4k}}$.Coefficient C_2 can be expressed by equating the stress boundary condition at the junction of damaged and undamaged zone i.e.$\sigma_{rr}|_{r=r_2} = -R_{23}$.Reactive force R_{23} can be evaluated by the displacement continuity condition at the interface.Detail procedure is alike as described next for the void case in Section.2.7.2.

2.7.2 Case of Void($\sigma_{rr}|_{r=0} = P$)

We will give the expressions for stress,strain for the Heun Equation which analyze a system close to the real system.Corresponding expressions for the Kummer Differential Equation can be derived in the same way.

2.7.2.1 Displacement,Stress and Strain Field for Damaged Zone.

Hoop Strain(Using Equation:B.5):

$$\varepsilon_{\theta\theta} = C1\text{HeunB}\left(\sqrt{5-4k}, \sqrt{2}, 1+2k, -2k\sqrt{2}, \frac{1}{2}\frac{\sqrt{2}r_1}{r}\right) r^{\frac{1}{2}-\frac{1}{2}\sqrt{5-4k}} + C2\text{HeunB}\left(-\sqrt{5-4k}, \sqrt{2}, 1+2k, -2k\sqrt{2}, \frac{1}{2}\frac{\sqrt{2}r_1}{r}\right) r^{\frac{1}{2}+\frac{1}{2}\sqrt{5-4k}} \tag{2.39}$$

Radial Strain(Using Equation:B.4):

$$\varepsilon_{rr} = \left[r^{-\frac{1}{2}\sqrt{5-4k}-\frac{1}{2}} \sum C_1 t^1_{mn} + r^{\frac{1}{2}\sqrt{5-4k}-\frac{1}{2}} \sum C_2 t^2_{mn} \quad (m = 1, 2 \quad ; \quad n = 1\ldots2)\right] \tag{2.40}$$

Where,

$$t^1_{11} = \frac{1}{r}\left(-\frac{1}{2}\sqrt{5-4k} - \frac{1}{2}\right)\text{HeunB}\left(\sqrt{5-4k}, \sqrt{2}, 1+2k, -2k\sqrt{2}, \frac{1}{2}\frac{\sqrt{2}r_1}{r}\right) \tag{2.41}$$

$$t^1_{12} = -\frac{1}{2r^3}\sqrt{2}r_1\text{HeunBPrime}\left(\sqrt{5-4k}, \sqrt{2}, 1+2k, -2k\sqrt{2}, \frac{1}{2}\frac{\sqrt{2}r_1}{r}\right) \tag{2.42}$$

$$t^2_{21} = -\frac{1}{2r^2}\sqrt{2}r_1\text{HeunBPrime}\left(-\sqrt{5-4k}, \sqrt{2}, 1+2k, -2k\sqrt{2}, \frac{1}{2}\frac{\sqrt{2}r_1}{r}\right) \tag{2.43}$$

$$t^2_{22} = -\frac{1}{2r^3}\left(\frac{1}{2}\sqrt{5-4k} - \frac{1}{2}\right)\text{HeunB}\left(-\sqrt{5-4k}, \sqrt{2}, 1+2k, -2k\sqrt{2}, \frac{1}{2}\frac{\sqrt{2}r_1}{r}\right) \tag{2.44}$$

Hoop Stress(Using Equation:B.2):

$$\sigma_{\theta\theta} = \frac{E\left(1-d(r)\right)}{(1-k^2)}\left(\varepsilon_{\theta\theta} + k\varepsilon_{rr}\right) \tag{2.45}$$

Radial Stress(Using Equation:B.3):

$$\sigma_{rr} = \frac{E\left(1-d(r)\right)}{(1-k^2)}\left(k\varepsilon_{\theta\theta} + \varepsilon_{rr}\right) \tag{2.46}$$

Here $d(r)$ is the damage index.At $r = r_1, d(r) = 1$, hence the stress is zero.

2.7.2.2 Boundary Condition and reactive Force

Boundary Condition: Ref.to fig(2.1),we write the boundary condition for the damaged zone as:

$$\begin{cases} \text{At } r = r_1 & : \quad \sigma_{rr} = 0 \\ \text{At } r = r_2 & : \quad \sigma_r r = -R_{23} \end{cases} \tag{2.47}$$

Using the Boundary Condition(2.47),we can derive the coefficients C_1,C_2 and R_{23}[8].

Coefficients C_1,C_2:

$$C_1 = R_{23}\frac{C_{21}}{-C_{21}C_{12} + C_{22}C_{11}} \tag{2.48}$$

$$C_2 = R_{23}\frac{C_{11}}{-C_{21}C_{12} + C_{22}C_{11}} \tag{2.49}$$

Here C_{11},C_{21},C_{12},C_{22} can be expressed as:

$$C_{11} = \frac{E(1-d(r))}{1-k^2} r_1^{-\frac{3}{2}-\frac{1}{2}\sqrt{5-4k}\sum t^i_{C_{11}} \quad i=1\cdots 3} \tag{2.50}$$

$$t^1_{C_{11}} = k\text{HeunB}\left(\sqrt{5-4k}, \sqrt{2}, 1+2k, -2k\sqrt{2}, \frac{1}{2}\sqrt{2}\right) \tag{2.51}$$

$$t^2_{C_{11}} = \left(-\frac{1}{2}\sqrt{5-4k} - \frac{1}{2}\right)\text{HeunB}\left(\sqrt{5-4k}, \sqrt{2}, 1+2k, -2k\sqrt{2}, \frac{1}{2}\sqrt{2}\right) \tag{2.52}$$

$$t^3_{C_{11}} = -\frac{1}{\sqrt{2}}\text{HeunBPrime}\left(\sqrt{5-4k}, \sqrt{2}, 1+2k, -2k\sqrt{2}, \frac{1}{2}\sqrt{2}\right) \tag{2.53}$$

$$C_{21} = \frac{E(1-d(r))}{1-k^2} r_1^{-\frac{3}{2}+\frac{1}{2}\sqrt{5-4k}\sum t^i_{C_{21}} \quad i=1\cdots 3} \tag{2.54}$$

$$t^1_{C_{21}} = k\text{HeunB}\left(-\sqrt{5-4k}, \sqrt{2}, 1+2k, -2k\sqrt{2}, \frac{1}{2}\sqrt{2}\right) \tag{2.55}$$

$$t^2_{C_{21}} = \left(\frac{1}{2}\sqrt{5-4k} - \frac{1}{2}\right)\text{HeunB}\left(-\sqrt{5-4k}, \sqrt{2}, 1+2k, -2k\sqrt{2}, \frac{1}{2}\sqrt{2}\right) \tag{2.56}$$

$$t^3_{C_{21}} = -\frac{1}{\sqrt{2}}\text{HeunBPrime}\left(-\sqrt{5-4k}, \sqrt{2}, 1+2k, -2k\sqrt{2}, \frac{1}{2}\sqrt{2}\right) \tag{2.57}$$

$$C_{12} = \frac{E(1-d(r))}{1-k^2} r_2^{-\frac{3}{2}-\frac{1}{2}\sqrt{5-4k}\sum t^i_{C_{12}} \quad i=1\cdots 3} \tag{2.58}$$

$$t^1_{C_{12}} = k\text{HeunB}\left(\sqrt{5-4k}, \sqrt{2}, 1+2k, -2k\sqrt{2}, \frac{r_1\sqrt{2}}{2r_2}\right) \tag{2.59}$$

$$t^2_{C_{12}} = \left(-\frac{1}{2}\sqrt{5-4k} - \frac{1}{2}\right)\text{HeunB}\left(\sqrt{5-4k}, \sqrt{2}, 1+2k, -2k\sqrt{2}, \frac{r_1\sqrt{2}}{2r_2}\right) \tag{2.60}$$

[8] Detail Derivation has been given in Appendix:D

$$t^3_{C_{12}} = -\frac{1}{\sqrt{2r_2}}\sqrt{2}r_1\text{HeunBPrime}\left(\sqrt{5-4k}, \sqrt{2}, 1+2k, -2k\sqrt{2}, \frac{r_1\sqrt{2}}{2r_2}\right) \tag{2.61}$$

$$C_{22} = \frac{E(1-d(r))}{1-k^2} r_2^{-\frac{3}{2}+\frac{1}{2}\sqrt{5-4k}} \sum t^i_{C_{22}} \quad i=1\cdots3 \tag{2.62}$$

$$t^1_{C_{22}} = k\text{HeunB}\left(-\sqrt{5-4k}, \sqrt{2}, 1+2k, -2k\sqrt{2}, \frac{r_1\sqrt{2}}{2r_2}\right) \tag{2.63}$$

$$t^2_{C_{22}} = \left(\frac{1}{2}\sqrt{5-4k} - \frac{1}{2}\right)\text{HeunB}\left(-\sqrt{5-4k}, \sqrt{2}, 1+2k, -2k\sqrt{2}, \frac{r_1\sqrt{2}}{2r_2}\right) \tag{2.64}$$

$$t^3_{C_{22}} = -\frac{1}{\sqrt{2r_2}}\sqrt{2}r_1\text{HeunBPrime}\left(-\sqrt{5-4k}, \sqrt{2}, 1+2k, -2k\sqrt{2}, \frac{r_1\sqrt{2}}{2r_2}\right) \tag{2.65}$$

1. **Reactive Force R_{23} for Stress is specified at the outer boundary($\sigma_{rr}|_{r=r_3} = -P$)**

$$R_{23} = \frac{f_P}{C_{R_p} - U_{r_2}} \tag{2.66}$$

$$f_P = 2\frac{Pr_2r_3^2}{E\left(r_3^2 - r_2^2\right)} \tag{2.67}$$

$$C_{R_p} = \frac{(1+\nu)r_2r_3^2 + (1-\nu)r_2^3}{E\left(r_3^2 - r_2^2\right)} \tag{2.68}$$

$$U_{r_2} = c_{C_1}\text{HeunB}\left(\sqrt{5-4k}, \sqrt{2}, 1+2k, -2k\sqrt{2}, \frac{1}{2}\frac{\sqrt{2}r_1}{r_2}\right) r_2^{-\frac{1}{2}-\frac{1}{2}\sqrt{5-4k}} + c_{C_2}\text{HeunB}\left(-\sqrt{5-4k}, \sqrt{2}, 1+2k, -2k\sqrt{2}, \frac{1}{2}\frac{\sqrt{2}r_1}{r_2}\right) r_2^{-\frac{1}{2}+\frac{1}{2}\sqrt{5-4k}} \tag{2.69}$$

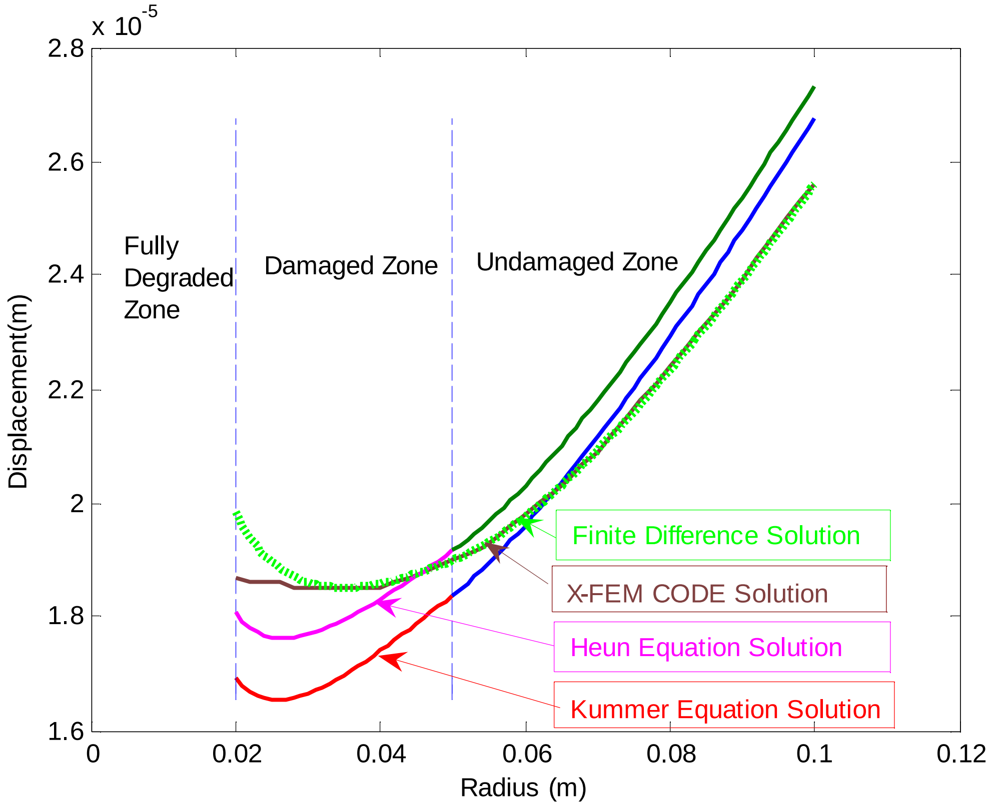
$$\text{Here,} \quad c_{C_1} = \frac{C_{21}}{-C_{21}C_{12}+C_{22}C_{11}} \quad c_{C_2} = \frac{C_{12}}{-C_{21}C_{12}+C_{22}C_{11}} \tag{2.70}$$

2. **Reactive Force R_{23} Displacement is specified at the outer boundary($u_r|_{r=r_3} = -U$)**

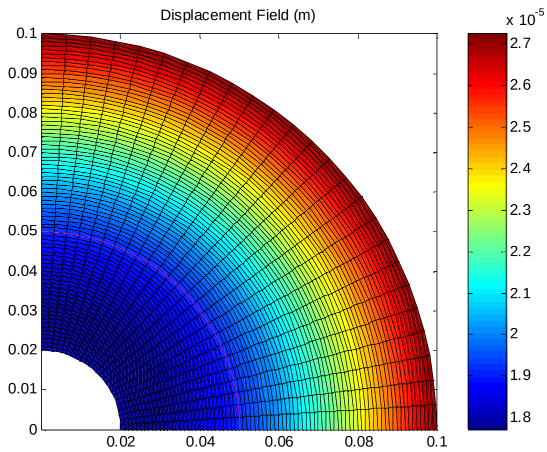
$$R_{23} = \frac{f_U}{C_{R_u} - U_{r_2}} \quad \text{Where,} \quad f_U = 2\frac{Ur_2r_3}{\left(r_3^2(1-\nu)+r_2^2(1+\nu)\right)} \quad C_{R_u} = \frac{r_2\left(-1+\nu^2\right)\left(-r_3^2+r_2^2\right)}{E\left(r_3^2(1-\nu)+r_2^2(1+\nu)\right)} \tag{2.71}$$

2.8 Numerical Result and Discussion

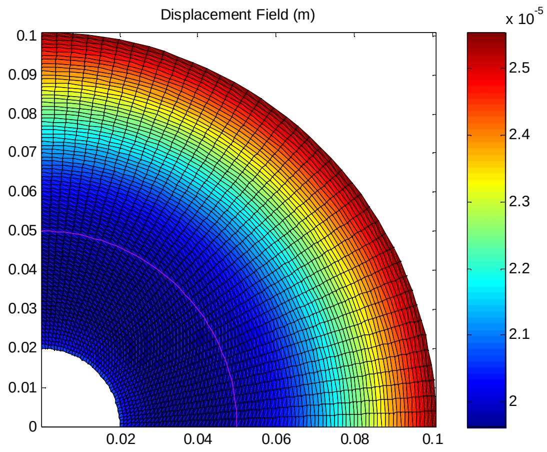
: We consider the same numerical values as for Hypergeometric Differential Equation i.e.$l_c = 0.03$m, $r_2 = 0.05$m, $r_3 = 0.1$m, $E = 37.7\times10^9 \frac{N}{m^2}$, $\nu = 0.20$, $P = 1\times10^7 \frac{N}{m^2}$.We are considering the case of Plane Stress $k = \nu$.For Plane Strain,we need to change the Poisson Ration.



(a) Displacement w.r.t. Radius

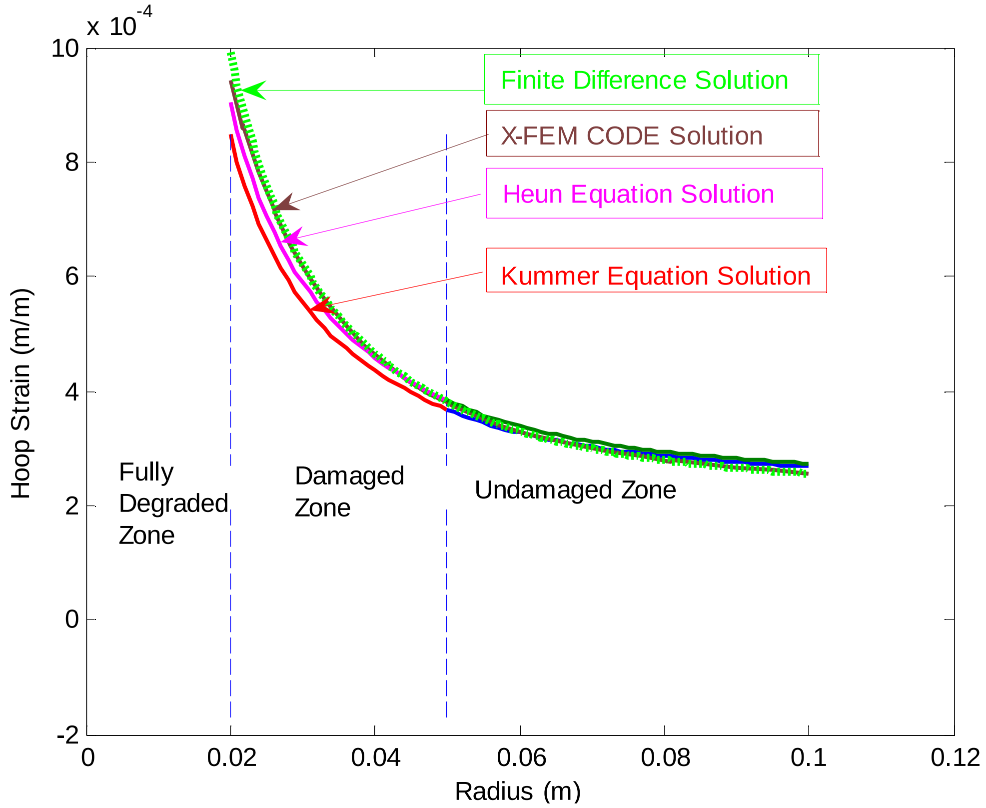


(b) Heun Solution

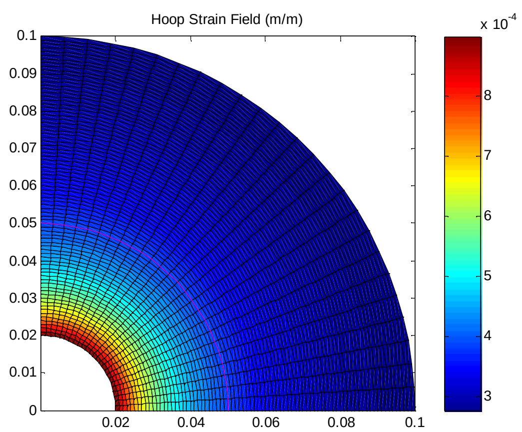


(c) Finite Difference Solution

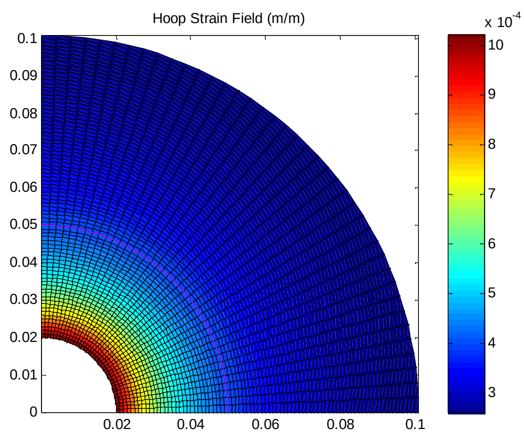
FIGURE 2.4: (a)Comparison of displacement with computational(XFEM Code and Finite Difference Solution) and analytical solution(Kummer and Heun Equation).(b)Heun Equation Solution (c)Finite Difference Solution



(a) Hoop Strain w.r.t. Radius



(b) Heun Solution



(c) Finite Difference Solution

FIGURE 2.5: (a)Comparison of Hoop Strain with computational(XFEM Code and Finite Difference Solution) and analytical solution(Kummer and Heun Equation).(b)Heun Equation Solution (c)Finite Difference Solution

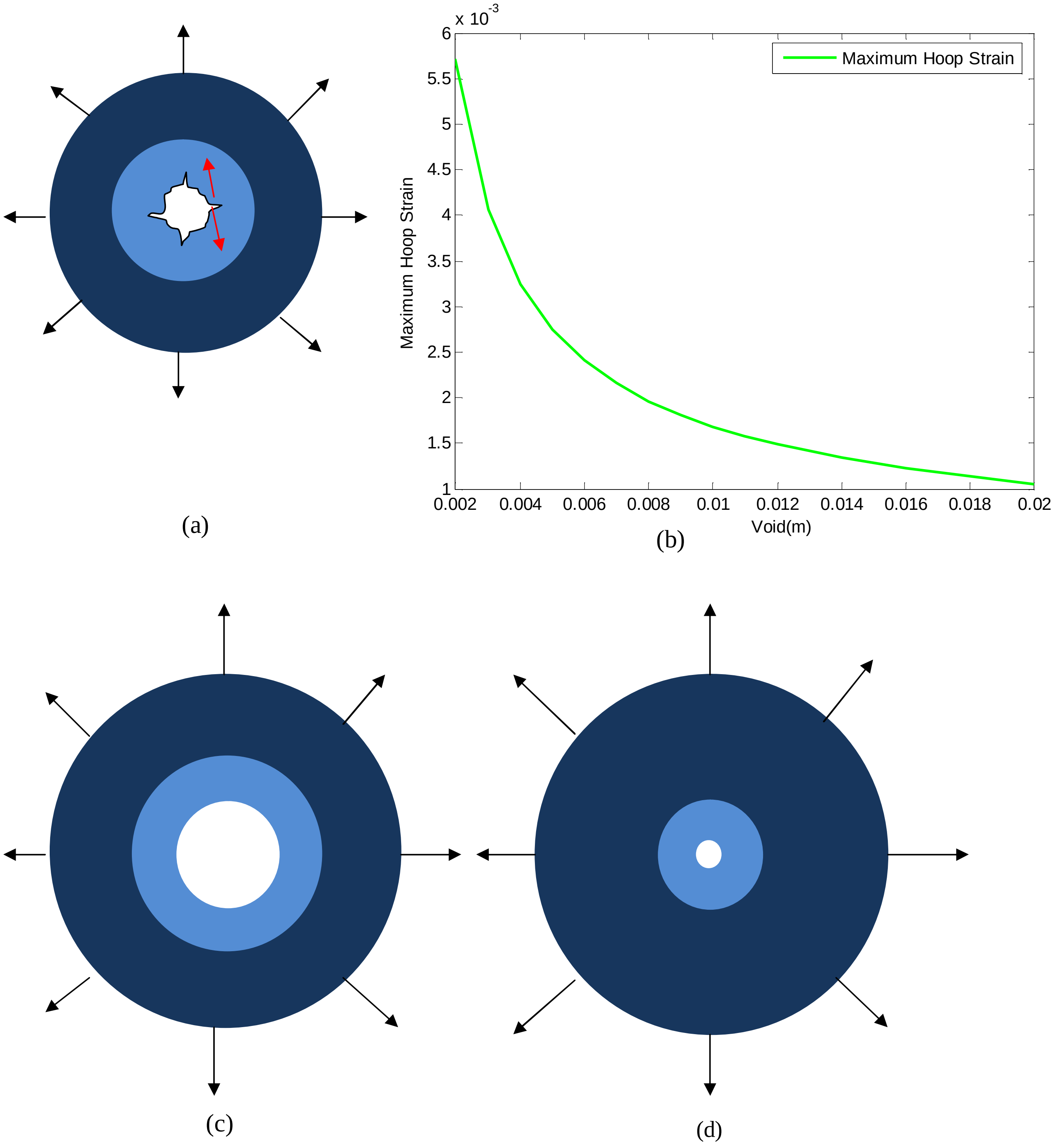


FIGURE 2.6: Variation of maximum value of Hoop Strain for different values of radius of void(r_1).For void zone with less raidus ,the maximum hoop strain is more.It means void with less radius is more prone to crack initiation.

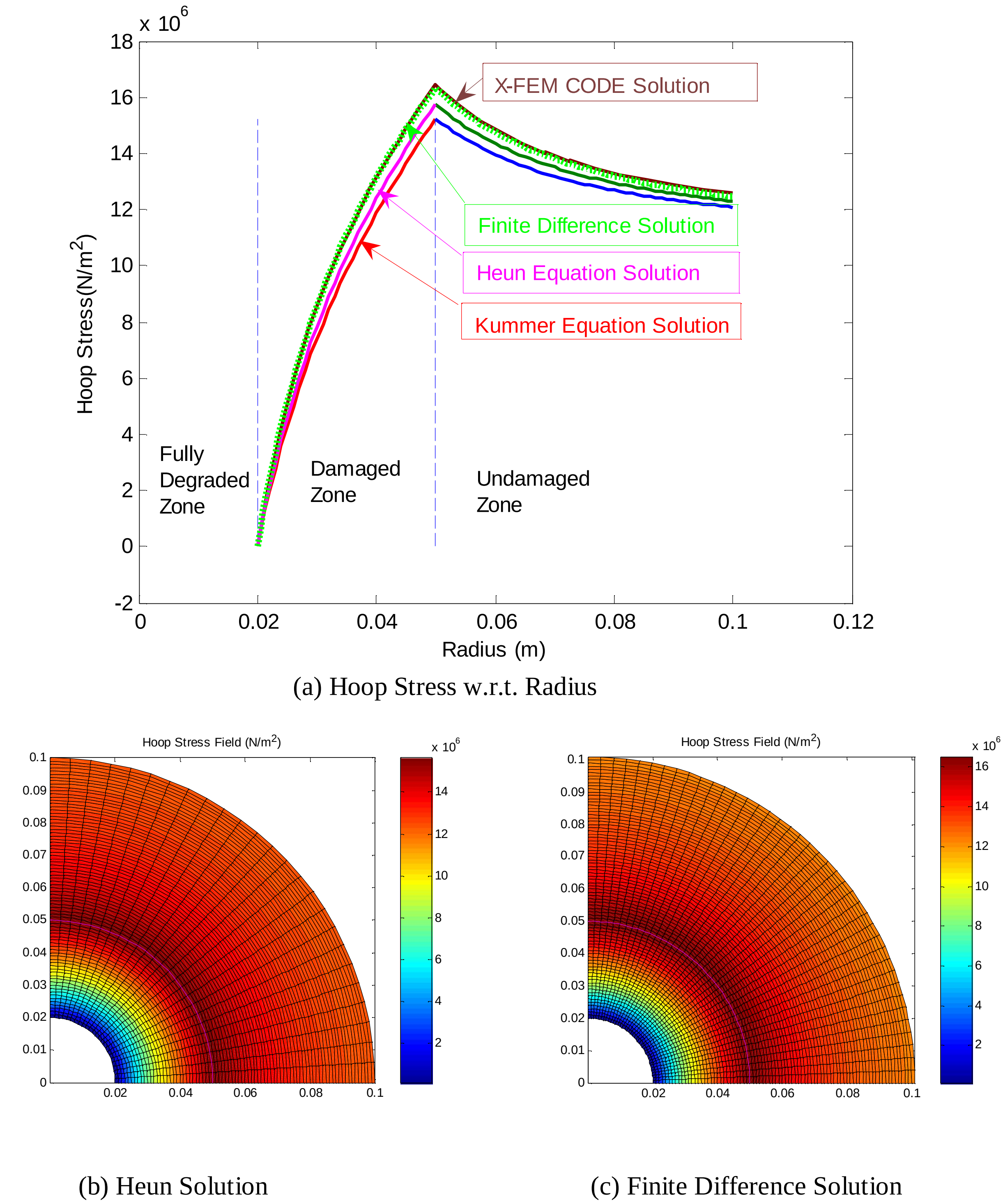
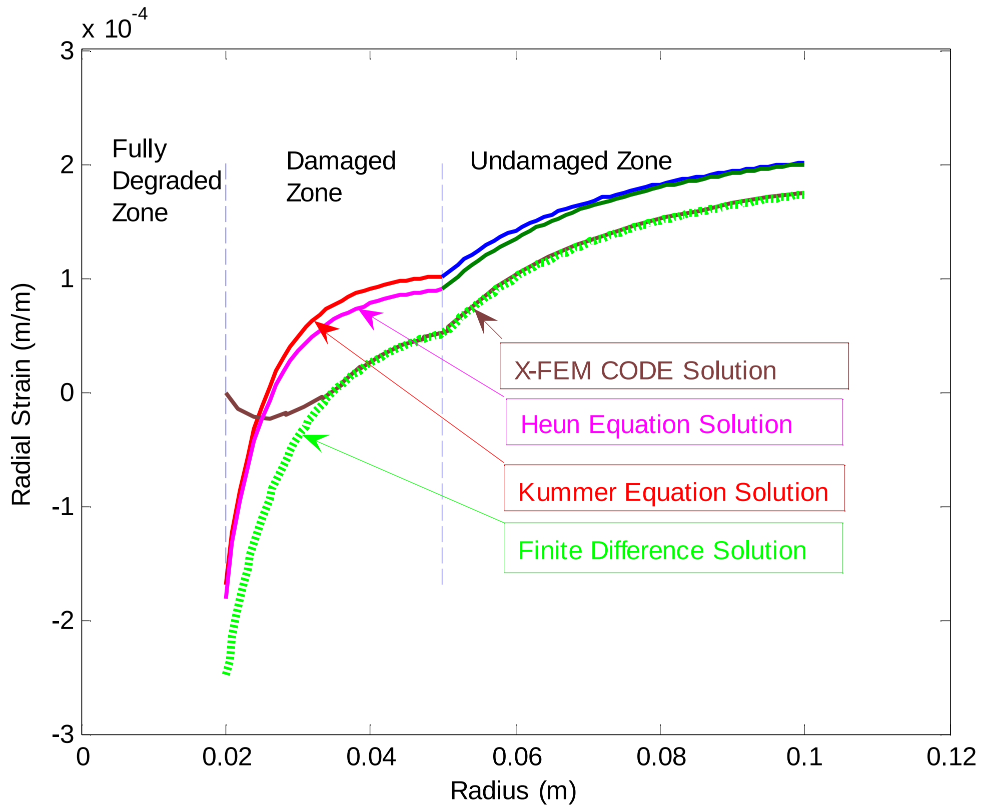
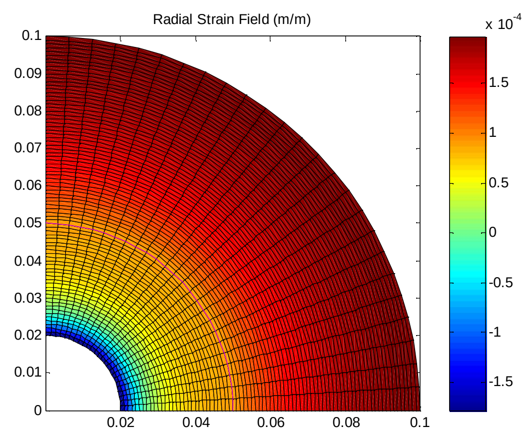


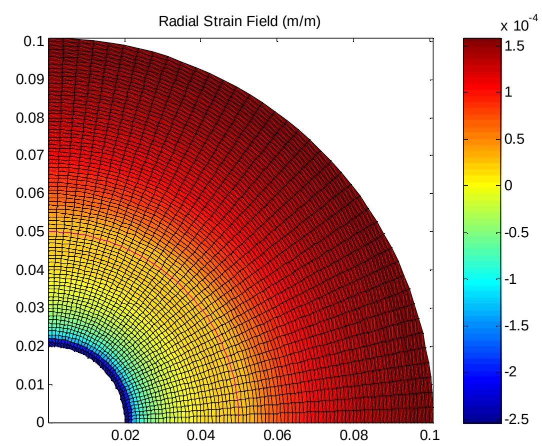
FIGURE 2.7: (a)Comparison of Hoop Stress with computational(XFEM Code and Finite Difference Solution) and analytical solution(Kummer and Heun Equation).(b)Heun Equation Solution (c)Finite Difference Solution



(a) Radial Strain w.r.t. Radius

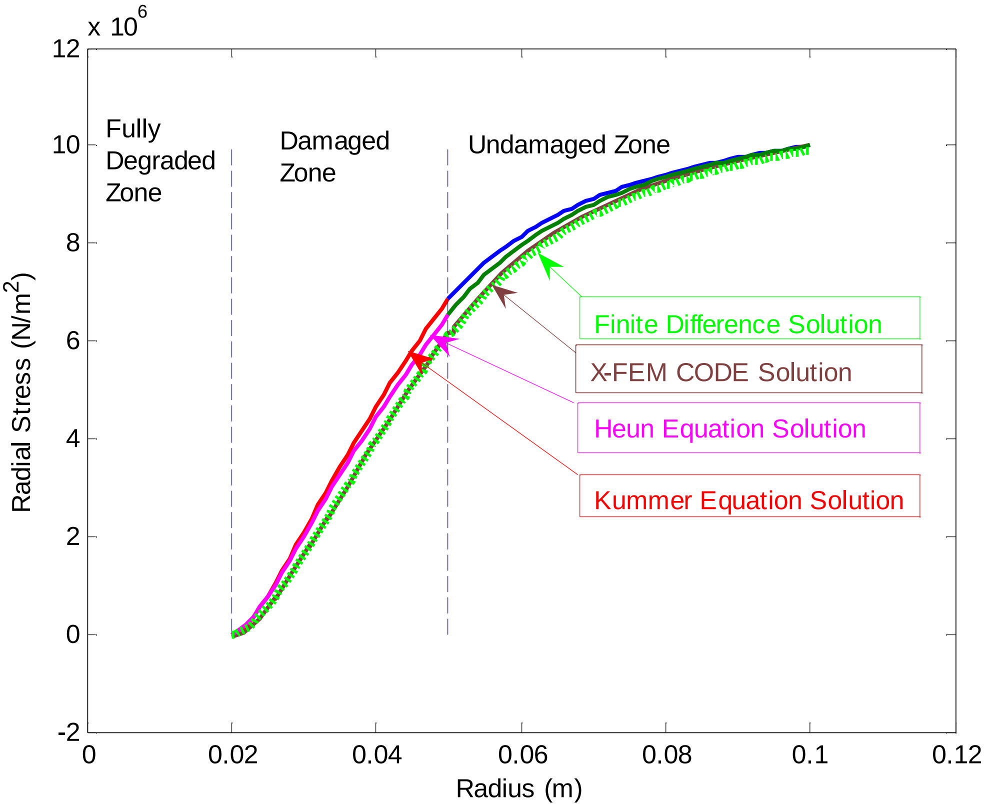


(b) Heun Solution

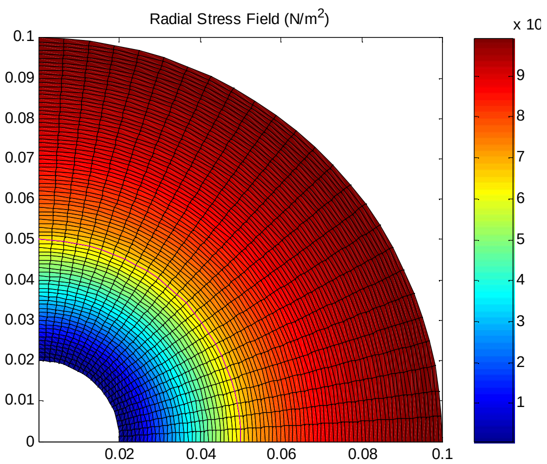


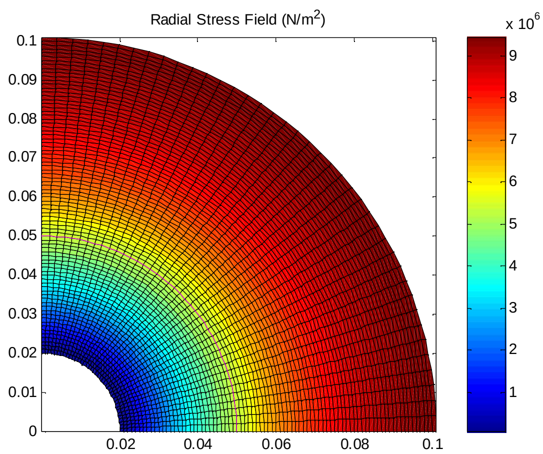
(c) Finite Difference Solution

FIGURE 2.8: (a)Comparison of Radial Strain with computational(XFEM Code and Finite Difference Solution) and analytical solution(Kummer and Heun Equation).(b)Heun Equation Solution (c)Finite Difference Solution



(a) Radial Stress w.r.t. Radius





(b) Heun Solution

(c) Finite Difference Solution

FIGURE 2.9: (a)Comparison of Radial Stress with computational(XFEM Code and Finite Difference Solution) and analytical solution(Kummer and Heun Equation).(b)Heun Equation Solution (c)Finite Difference Solution

Discussion :

Ref.Fig:2.4,we observe that Finite Difference Solution matches well with the computational solution.At the neighbourhood of $r = r_1$,there is slight mismatch.May be that is due the reason that coarse mesh has been used for the computation using X-FEM code.Analytical solution (Heun and Kummer solution) finds their places in the neighbourhood of the computational solution with Heun solution better as we have considered $\left(\frac{r_1}{r}\right)^2$ term for that case.In all cases, we can observe that the minimum displacement is not exactly at $r = r_1$.

Ref.Fig.2.5,we get the variation of Hoop Strain with radius.Computational and analytical solution matches very well.From Fig.2.6(b),we observe the variation of maximum Hoop Strain with radius of void zone.Hoop Strain is the prime cause of inducing crack (shown red in Fig.2.6(a)).We can observe that a system with less inner radius of void(Fig.2.6(d)) is more prone to initiation of crack as compared to system with more void zone(Fig.2.6(c)).

In Fig.2.7,we get the variation of Hoop Stress with radius.Hoop Stress is maximum at the junction between damaged and undamaged zone and drops to zero at the inner free surface.Finite Difference Solution matches exactly with the analytical solution very close.

Ref.Fig.2.8,we get the variation of Radial Strain with radius.Finite Difference solution matches with the XFEM code except in the region close to $r = r_1$.X-FEM code gives the solution of radial strain at $r = r_1$ as zero.However, we compute $\frac{du}{dr}|_{r=0}$ using discretization scheme.Since,computed displacement at two neighbouring points is not the same,$\frac{du}{dr}|_{r=0}$ is not zero at $r = r_1$.In Fig.2.5,we observe that hoop strain is continuous with same slope at the junction between damaged and undamaged system.However,radial strain is not continuous with same slope.

Ref.to Fig.2.9,Computational solution matches and the analytical solutions are closed to the computational solution.Unlike hoop stress, radial stress is maximum at the outer raidus and drops to zero at the inner free surface.

2.9 Chapter Summary

In summary,in this Chapter,we have developed Equilibrium and Compatibility equation for the damaged zone.We have developed numerical solution using Finite Difference Method for the system with damaged and undamaged zone.We have extended our studies developing analytical solution.We have analysed that for the void case,we need to kill the singularity by doing some approximation in order to get the analytical solution.We have compared our analytical and semi-analytical solution with the solution from the XFEM code.

Chapter 3

Asymptotic Analysis of Displacement, Stress and Strain Field for Void Zone Tending Zero($d = 1$ at the center)

3.1 Introduction

In Chapter(2),we have developed the analytical solution for displacement, stress and strain field for a circular plate having void at middle and with a transition zone.In this chapter,we will extend our study of analytical solution for the case when the radius of the void tends to zero. We will study the governing equation for stress, strain and displacement and analyze the asymptotic behavior as void zone tends to zero.The case of void zone tending zero ($d = 1$ at the center) is a very realistic case. The young modulus of the material in the transition zone is given by $E_{dam} = E_{undam}\,(1 - d(r))$.The damage index is 1 only at the midpoint i.e. the material is having zero resistance at the midpoint. Or we can say the structure has a void at mid point.E is continuous at the end of the transition zone. Hence all fields of interest are continuous. However, it is of particular interest to know the behavior of the fields of interest (displacement, stress and strain) as the void zone tends to zero i.e.(Ref.Fig 2.1)

$$\lim_{\frac{r}{r_1} \to 1, r_1 \to 0} \{u, \sigma, \varepsilon\} = ? \tag{3.1}$$

3.2 Motivation Behind Asymptotic Analysis

It is of very much practical interest to evaluate the mechanical behaviour of the fields of interest and their orders for the siuation when material just starts fully degrading i.e. the damage index $d = 1$ only at the center.As we have shown in Fig.2.6 that maximum hoop strain occurs at the inner radius and it is the prime cause of inducing crack.Less is the radius of the void , more is the maximum hoop strain.Hence as the material at the center starts degrading,next step is crack initiation and growth.Zhao et al [19] has studied asymptotically the fatigue crack growth based on Damage Mechanics.In the traditional system,crack initiation and crack growth are treated respectively as two independent material deterioration process.But with the introduction and developement of Continuum Damage Mechanics [20][21],it is now more and more possible for us to study thease two phenomena within one unified field.

3.2.1 Erroneous Solution by Cauchy-Euler Equation Approch

At a first look, we might get the impression that this problem can be treated in an easy platform as the problem of no void case.Hence we can use the general differential equation governing displacement and proceed with a damage index having a value 1 at the midpoint and 0 at the end of the transition zone. However as shown below, this approach will not produce the desired result and the required situation of void zone tending to zero is not captured.

Ref.Fig 3.1,we consider the circular plate subjected to axisymmetric loading having the following zones:

$$\begin{cases} \text{d}=1-\frac{r}{r_2} & : \text{Zone2}\{\text{Damaged Zone(Radius } r=r_2)\} \\ \text{d}=0 & : \text{Zone3}\{\text{Undamaged Zone(Inner Radius } r=r_2 \text{ and Outer Radius } r=r_3)\} \end{cases} \tag{3.2}$$

3.2.2 Solution for Damaged Zone

We rewrite the Equilibrium Equation for damaged zone(Equation 3.2.2) as derived in Section 2.3.2 to get the analytical expressions for displacement ,stress and strain for the damaged zone[1].

$$\frac{\partial^2 u}{\partial r^2} + \beta_1(r)\frac{\partial u}{\partial r} + \beta_0(r)u = 0$$

[1] The solution for undamaged zone is same as described in Section 2.2

Substituting the expression of damage index d as described in Equation 3.2 in the above equation,we obtain the following **Cauchy-Euler or Equidimensional Equation**:

$$r^2\frac{\partial^2 u}{\partial r^2} + 2r\frac{\partial u}{\partial r} - (1-\nu)\,u = 0 \tag{3.3}$$

We obtain the solution:

$$u = c_{11}r^{\theta_1} + \frac{c_{12}}{r^{\theta_2}} \tag{3.4}$$

Where:

$$\left|\begin{array}{lcl} \theta_1 &=& \frac{1}{2}\left(-1+\sqrt{1+4(1-\nu)}\right) > 0 \\ \theta_2 &=& \frac{1}{2}\left(1+\sqrt{1+4(1-\nu)}\right) > 0 \end{array}\right.$$

At$r = 0$,$u = 0$,Hence we take $c_{12} = 0$.The Equation 3.4 reduces to $u = c_{11}r^{\theta_1}$.
The radial and transverse stresses are obtained as:

$$\sigma_{rr} = c_{11}\frac{E}{(1-\nu^2)}\left[\nu + \frac{1}{2}\left(-1+\sqrt{1+4(1-\nu)}\right)\right] r^{\frac{1}{2}\left(-3+\sqrt{1+4(1-\nu)}\right)} \tag{3.5}$$

$$\sigma_{\theta\theta} = c_{11}\frac{E}{(1-\nu^2)}\left[1 + \frac{\nu}{2}\left(-1+\sqrt{1+4(1-\nu)}\right)\right] r^{\frac{1}{2}\left(-3+\sqrt{1+4(1-\nu)}\right)} \tag{3.6}$$

Equating the displacemnet for both the zones at the interface(i.e.at $r = r_2$) and using the Boundary Condition $\sigma_{rr}|_{r=r_2} = -R_{23}$,we get the coefficient c_{11} and Reactive force R_{23} as[2]:

$$R_{23} = -\frac{f_U\left[1+\frac{\nu}{2}\left(-1+\sqrt{1+4(1-\nu)}\right)\right] r_2^{\frac{1}{2}\left(-3+\sqrt{1+4(1-\nu)}\right)}}{r_2^{\frac{1}{2}\left(-3+\sqrt{1+4(1-\nu)}\right)}\left(1+\left[1+\frac{\nu}{2}\left(-1+\sqrt{1+4(1-\nu)}\right)\right] C_{R_u}\right)} \tag{3.7}$$

$$c_{11} = \frac{R_{23}}{\left[1+\frac{\nu}{2}\left(-1+\sqrt{1+4(1-\nu)}\right)\right] r_2^{\frac{1}{2}\left(-3+\sqrt{1+4(1-\nu)}\right)}} \tag{3.8}$$

Here f_U and C_{R_u} can be obtained from Equation:2.71.

3.2.3 Numerical Result and Conclusion

We consider the following parameters to perform the analysis: $r_1 = 0.01$m, $r_2 = 0.04$m, $r_3 = 0.1$m, $E = 2\times10^{11}\frac{N}{m^2}$, $\nu = 0.30$, $U = -0.001$m.

Conclusion

The displacement variation is different in damaged and undamaged zone. The derivative of displacement is discontinuous at the interface and hence stress and strain fields too. Moreover, from the expression of stress and strain field in inner domain, we find that at the center ($r = 0$), the stress and strain field are singular. Hence, we need to perform asymptotic

[2]Procedure is same as described in Appendix D for Circular plate having void

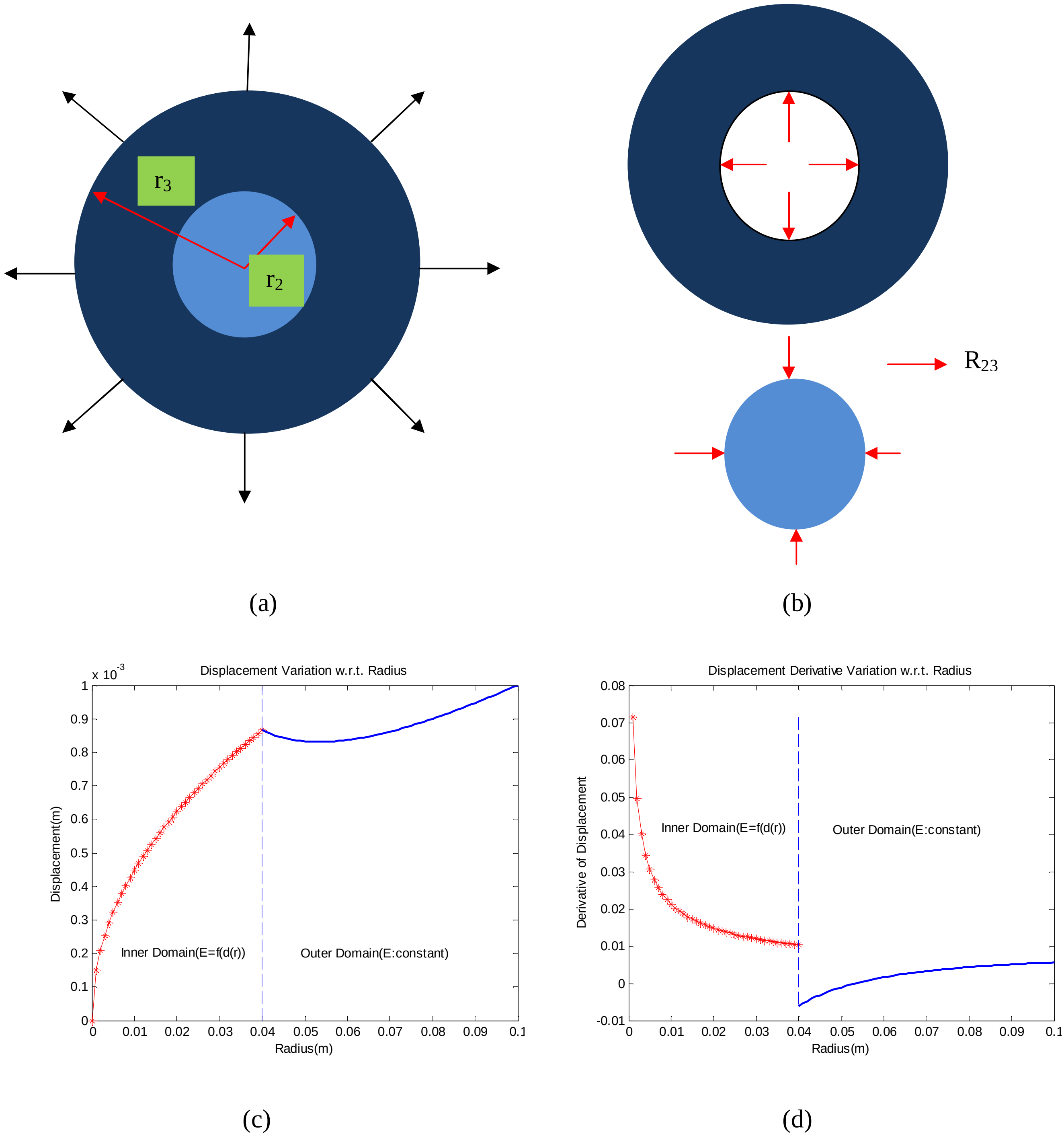


FIGURE 3.1: **ERRONEOUS RESULT** through Cauchy-Euler Equation Approach(a)Circular Plate with Zero Void(Inner radius as in Fig.2.1 is zero i.e.$r_1 = 0$) (b)Reactive force R_{23} at the interface (c) Displacement(m) variation w.r.t.Radius(m).Variation is not same in damaged and undamaged zone.(d)Displacement Derivative($\frac{\partial u}{\partial r}$) w.r.t. Radius(m).$\frac{\partial u}{\partial r}$ is discontinuous at the interface.

analysis in order to understand the behavior of fields of interest for the case void zone tends to zero.

3.3 Asymptotic Analysis for Void Zone Tending Zero

We consider the circular plat described in Chapter: 2 by Fig.2.1.The three zones of the problem are as described in Equation:2.1.
Heun Solution developed in Section.2.7 is the **accurate analytical solution** for the case when $r_2 = l_c$ i.e.damage index $d = 1$ only at the center.
We rewrite the Heun Solution given by Equation.2.38:

$$u = C_1 \text{HeunB}\left(\sqrt{5-4k}, \sqrt{2}, 1+2k, -2k\sqrt{2}, \frac{1}{2}\frac{\sqrt{2}r_1}{r}\right) r^{-\frac{1}{2}-\frac{1}{2}\sqrt{5-4k}} + C_2 \text{HeunB}\left(-\sqrt{5-4k}, \sqrt{2}, 1+2k, -2k\sqrt{2}, \frac{1}{2}\frac{\sqrt{2}r_1}{r}\right) r^{-\frac{1}{2}+\frac{1}{2}\sqrt{5-4k}} \tag{3.9}$$

From the above equation,we may get the impression that u is infinite at $r = 0$ because of the presence of the term $\frac{1}{r^{\frac{1}{2}+\frac{1}{2}\sqrt{5-4k}}}$.
From the expressions of stress and strain(Equation:2.39,2.40,2.45,2.46), we will get the impression that stress and strain also tends to infinite value. Therefore, we want to analyze the associated terms in the analytical solutions and investigate whether the displacement, stress and strain tends to zero, infinite or constant value as void zone tends to zero(i.e. $d = 1$ only at the center).

3.3.1 Displacement, Strain and Stress Field as Void Zone Tends To Zero

We note that we are considering the case when $r_1 \to 0$ and for any given k, we have $\text{HeunB}\left(\sqrt{5-4k}, \sqrt{2}, 1+2k, -2k\sqrt{2}, 0\right) = 1$ and $\text{HeunB}\left(-\sqrt{5-4k}, \sqrt{2}, 1+2k, -2k\sqrt{2}, 0\right) = 1$.HeunBPrime function alos tends to 1 as $r_1 \to 0$.

3.3.1.1 Coefficients C_1 and C_2

As described in Chapter: 2, coefficients C_1 and C_2 associated in the displacement,stress and strain terms can be expressed using Equation: 2.48 , 2.49 as:$C_1 = R_{23}\frac{C_{21}}{-C_{21}C_{12}+C_{22}C_{11}}$ and $C_2 = R_{23}\frac{-C_{11}}{-C_{21}C_{12}+C_{22}C_{11}}$.
Using Equation.2.51,2.52,2.53,we can prove that:

$$\lim_{r_1 \to 0} C_{11} \to -10^{\eta_1} \quad \text{Where} \quad \eta_1 > 1000 \tag{3.10}$$

Similarly,using Equation.2.55,2.56,2.57,we can prove that:

$$\lim_{r_1 \to 0} C_{21} \to -10^{\eta_2} \quad \text{Where} \quad \eta_2 \approx 0.2\eta_1 \tag{3.11}$$

From Equation.2.59,2.60,2.61 and Equation.2.63,2.64,2.65,we can observe that C_{12} and C_{22} becomes independent of r_1 as $r_1 \to 0$.Their values are negligible as compared to C_{11} and C_{21}. Using the above study,we can write that: It can be proved that:

$$\lim_{r_1 \to 0} C_1 = \lim_{r_1 \to 0} \frac{R_{23}}{-C_{12} + C_{22}\frac{C_{11}}{C_{21}}} \approx 10^{-0.80\eta_1} \approx 0 \tag{3.12}$$

$$\lim_{r_1 \to 0} C_2 = \lim_{r_1 \to 0} R_{23} \frac{-C_{11}}{-C_{21}C_{12} + C_{22}C_{11}} \approx -10^{-\beta} \quad \beta \approx 4 - 5 \tag{3.13}$$

3.3.1.2 Reactive Force R_{23}

We will investigate whether R_{23} goes to bounded or unbounded value as void zone tends to zero.

Case 1: Stress is specified at the outer boundary $(\sigma_{rr}|_{r=r_3} = -P)$ For this case,reactive force R_{23} is given by Equation: 2.66

$$R_{23} = \frac{f_P\,[\neq f(r, r_1)]}{C_{R_p}\,[\neq f(r, r_1)] - U_{r_2}\,[= f(r_1)]} \tag{3.14}$$

$$\lim_{r_1 \to 0} U_{r_2} = \underbrace{C_1 r_2^{-\frac{1}{2}\sqrt{5-4k}-\frac{1}{2}}}_{\approx 0} + \underbrace{C_2 r_2^{\frac{1}{2}\sqrt{5-4k}-\frac{1}{2}}}_{\text{Bounded value}} \tag{3.15}$$

$$\lim_{r_1 \to 0^+} R_{23} \cong \frac{f_P\,[\neq f(r, r_1)]}{C_{R_p}\,[\neq f(r, r_1)]} (\text{Bounded Value}) \tag{3.16}$$

Case 2: Displacement is specified at the outer boundary $(u_r|_{r=r_3} = -U)$
In the same way,for this case also it can be shown that R12 tends to bounded value as void zone tends to zero.

3.3.1.3 Displacement

From the equation of displacement(3.9)and using the analysis of two sections,we get:

$$\lim_{r_1 \to 0} u = \underbrace{C_1 r^{-\frac{1}{2}\sqrt{5-4k}-\frac{1}{2}}}_{\approx 0} + C_2 r^{\frac{1}{2}\sqrt{5-4k}-\frac{1}{2}} \to 0 \tag{3.17}$$

Comment: The displacement tends to zero quadratically proportional to $r_2^{\frac{1}{2}\sqrt{5-4k}-\frac{1}{2}}$ as void zone tends to zero.

3.3.1.4 Hoop Strain

From the equation of Hoop Strain(2.39) and using the above analysis, we can write:

$$\lim_{r\to r_1, r_1\to 0} \varepsilon_{\theta\theta} = \lim_{r\to r_1, r_1\to 0} \left(C_1 r^{-\frac{1}{2}\sqrt{5-4k}-\frac{3}{2}} + C_2 r^{\frac{1}{2}\sqrt{5-4k}-\frac{3}{2}} \right) \tag{3.18}$$

As $r_1 \to 0$ say $r_1 \approx 10^{-500}$,$C_1 \approx 10^{-1000}$,$r^{-\frac{1}{2}\sqrt{5-4k}-\frac{3}{2}} \approx 10^{1200}$,$C_2 \approx -10^{-4}$,$r^{\frac{1}{2}\sqrt{5-4k}-\frac{3}{2}} \approx 10^{200}$.Hence we observe that $\varepsilon_{\theta\theta}$ tends to large value as $r_1 \to 0$.

3.3.1.5 Radial Strain

Using the same logic and from the equation of Radial Strain(2.40),we can derive that ε_{rr} tends to large value as $r_1 \to 0$.

3.3.1.6 Hoop and Radial Stress

As damage index $d(r) = 1$ when $r_1 \to 0$,the radial and hoop stress is $\to 0$.

3.4 Numerical Result and Discussion

We consider the following numerical values: i.e. $l_c = 0.03$m, $r_2 = 0.03$m, $r_3 = 0.1$m, $E = 37.7 \times 10^9 \frac{N}{m^2}$, $\nu = 0.20$, $P = 1 \times 10^7 \frac{N}{m^2}$.

Discussion:

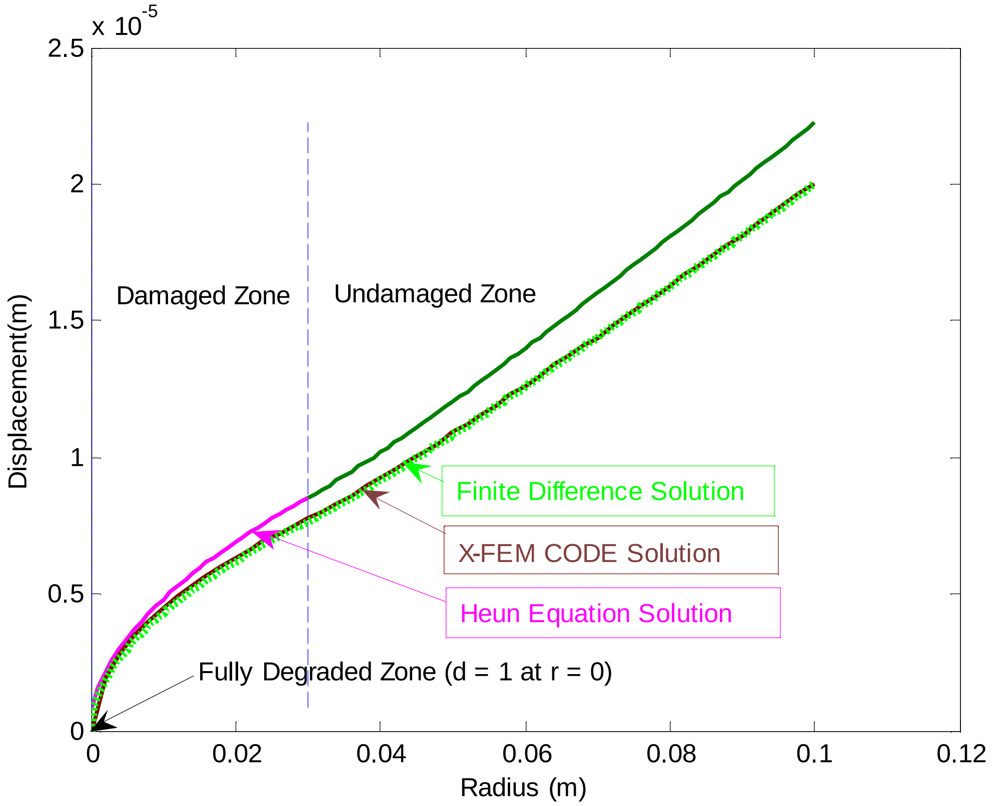
Ref.to Fig.3.2,Finite Difference solution matches exactly with the solution from X-FEM code.Heun solution matches well with the solution at the damaged zone and differs slightly at the junction point.Since the displacement is continuous with same slope at the junction,the Heun solution starts deviating form the computational solution in the undamaged zone.As shown in the analysis in the Section.3.3.1.3,the displacement tends to zero quadratically as $r_1 \to 0$.

From Fig.3.3,we observe that Hoop Strain tends to very large value as $r_1 \to 0$.Analytical and computational result matches well.

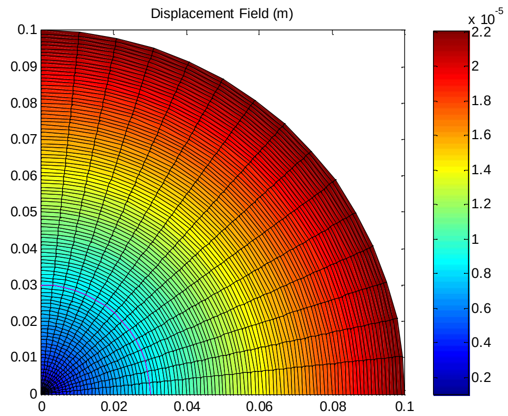
Ref.to Fig.3.4,Hoop Stress tends to zero as $r_1 \to 0$.Hoop Stress is maximum at the junction.All the solutions matches perfectly.

From Fig.3.5,we observe that radial strain tends to a large value as $r_1 \to 0$.

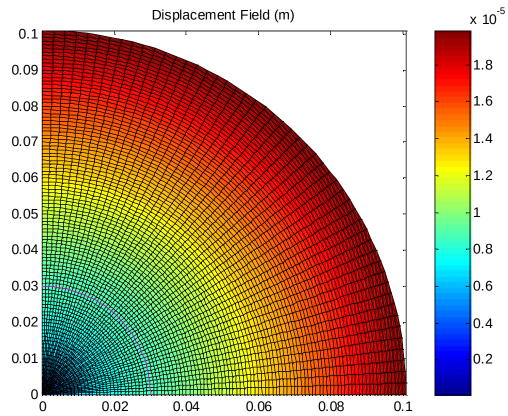
Ref.Fig.3.6,Radial stress tends to zero as $r_1 \to 0$.Radial stress is continous with same slope at the junction.Computational and Analytical solution matches perfectly.



(a) Displacement w.r.t. Radius

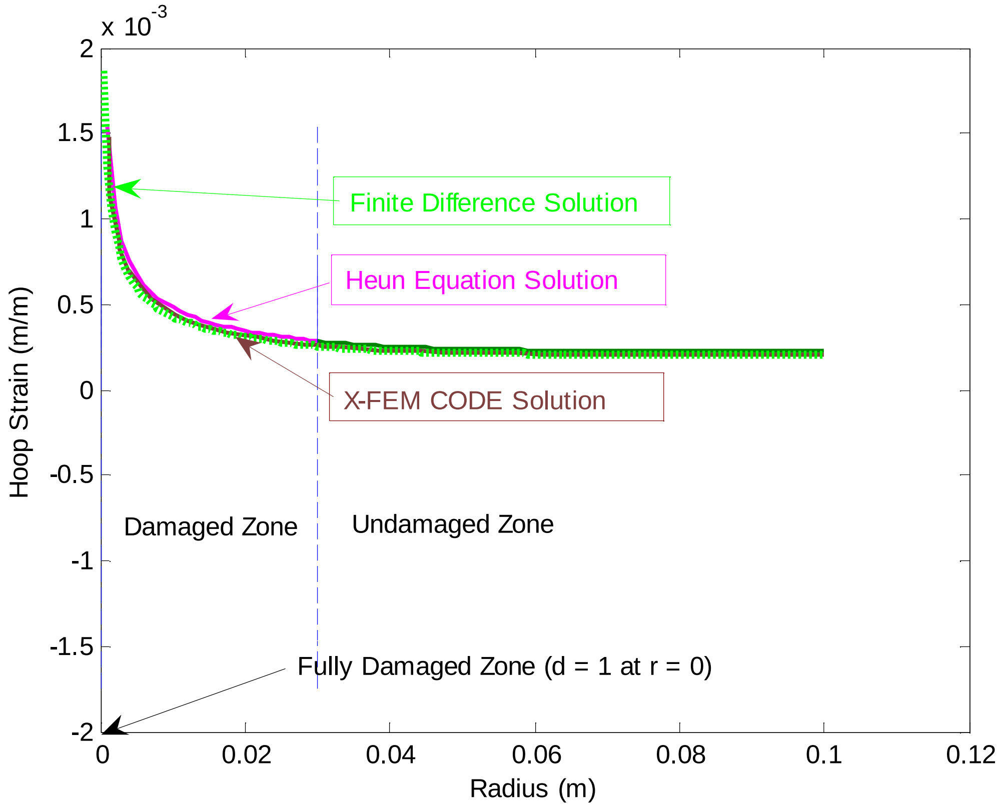


(b) Heun Solution

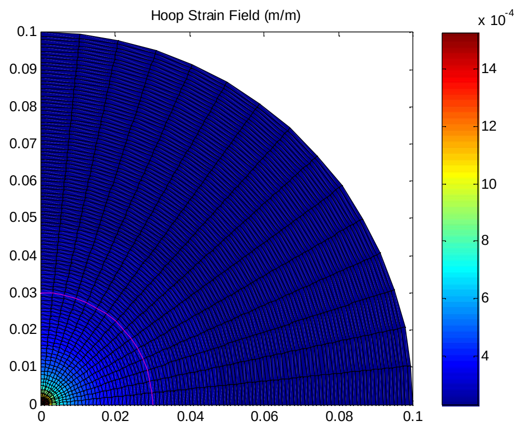


(c) Finite Difference Solution

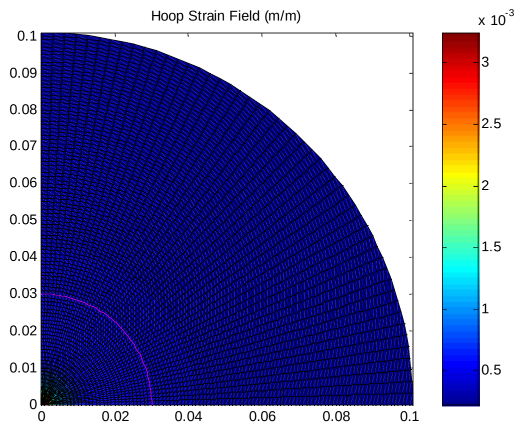
FIGURE 3.2: (a)Comparison of displacement with computational(XFEM Code and Finite Difference Solution) and analytical solution(Heun Equation).(b)Heun Equation Solution (c)Finite Difference Solution



(a) Hoop Strain w.r.t. Radius

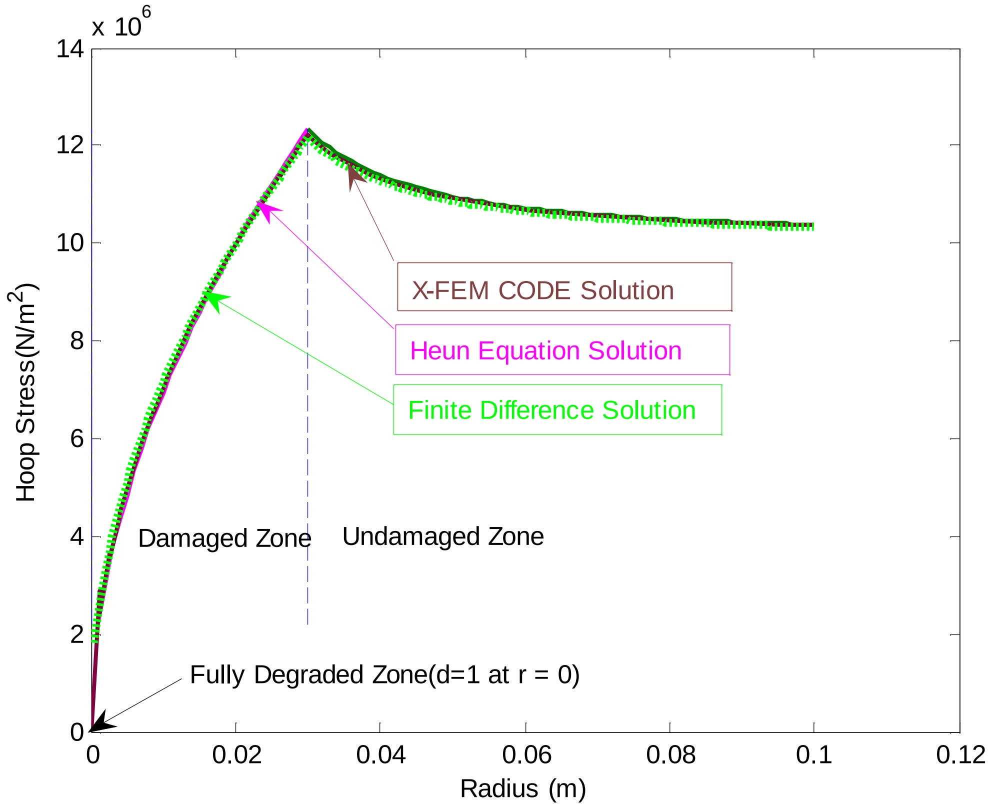


(b) Heun Solution

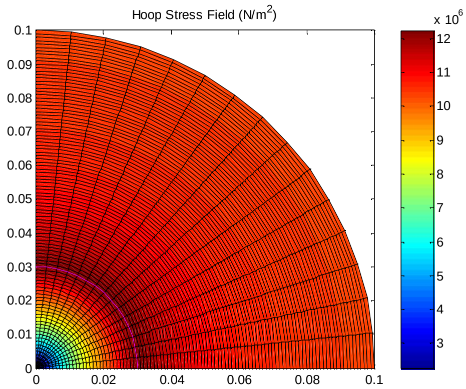


(c) Finite Difference Solution

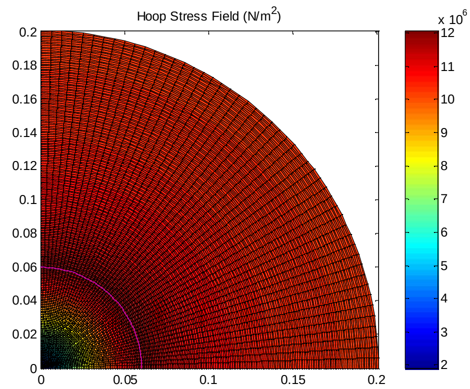
FIGURE 3.3: (a)Comparison of Hoop Strain with computational(XFEM Code and Finite Difference Solution) and analytical solution(Heun Equation).(b)Heun Equation Solution (c)Finite Difference Solution



(a)

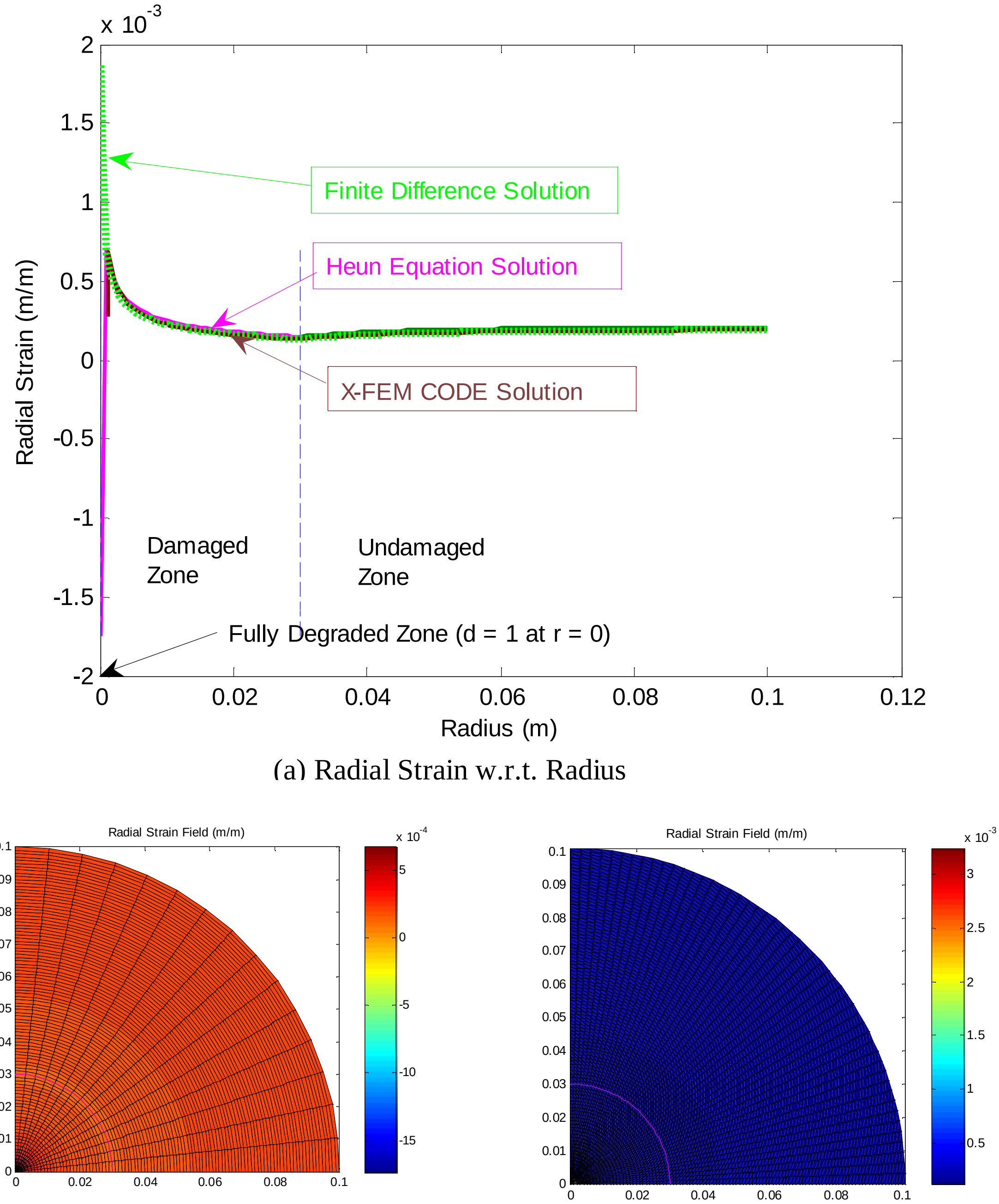


(b) Heun Solution



(c) Finite Difference Solution

FIGURE 3.4: (a)Comparison of Hoop Stress with computational(XFEM Code and Finite Difference Solution) and analytical solution(Heun Equation).(b)Heun Equation Solution (c)Finite Difference Solution

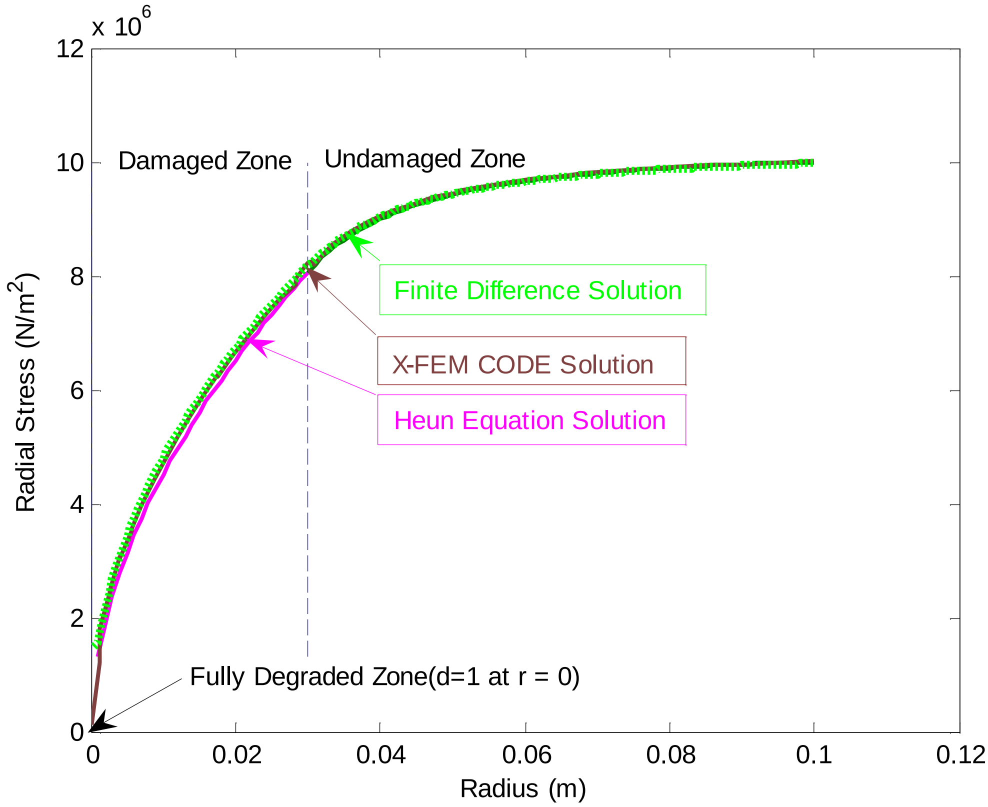


(a) Radial Strain w.r.t. Radius

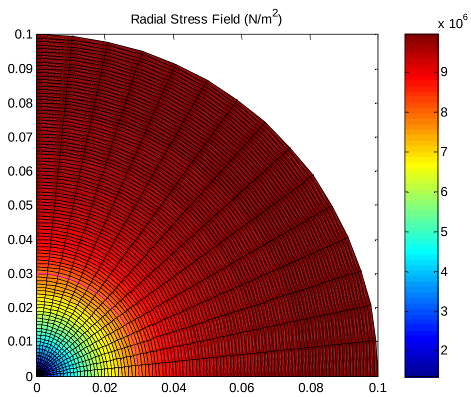
(b) Heun Solution

(c) Finite Difference Solution

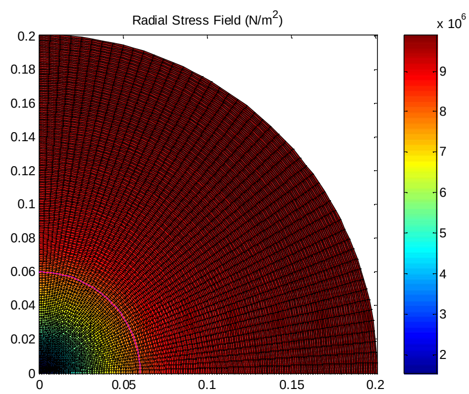
FIGURE 3.5: (a)Comparison of Radial Strain with computational(XFEM Code and Finite Difference Solution) and analytical solution(Heun Equation).(b)Heun Equation Solution (c)Finite Difference Solution



(a) Radial Stress w.r.t. Radius



(b) Heun Solution



(c) Finite Difference Solution

FIGURE 3.6: (a)Comparison of Radial Stress with computational(XFEM Code and Finite Difference Solution) and analytical solution(Heun Equation).(b)Heun Equation Solution (c)Finite Difference Solution

3.5 Chapter Summary

In summary,in this chapter,we have studied the behaviour of displacemnet,stress and strain field for the case when material just starts fully degrading at the center.We have investigated varous aspects of asymptotic analysis.We have analyzed the wrong approach by Equidimensional Equation which at first look seems to be the right approach for this case.We can also observe that for this case i.e.($d = 1 \quad \text{at} \quad r = 0$) the inner boundary condition for the no void case ($u|_{r=0} = 0$) and for the void case ($\sigma_{rr}|_{r=r_1} = 0$) coincides.We have observed that strain fields goes to large value while the displacement and stress field moves to zero.

Chapter 4

Analytical Force-Displacement Relationship Based on Growth Law in Thick Level Set Approach

4.1 Introduction

In Chapter 3, we have studied the asymptotic expression for the case when damage just starts at the center of the circular plate having a transition damage zone i.e $d = 1$ only at the center. The foundation of our study was the analytical expressions derived in Chapter 2.In this chapter, we will derive the analytical expression for the Force-Displacement relationship for 1D and for complex 2D case based on Growth Law in The Thick Level Set Approach developed by Prof.Nicolas MOËS at ECN,France[1]. We will study how the stiffness of the system varies with the damage propagation and how force responsible for damage propagation is related to corresponding displacement. We will use the analytical expressions derived in previous two chapters.

4.2 Force-Displacement Relationship for One-Dimensional Case

4.2.1 Analytical Expression of Force and Displacement

We first consider the $1D$ bar submitted to a proposed displacement U as shown in Fig.4.1(a).We assume that the damage will initiate at the left bar extremity.The damage zone will grow and its extent is denoted l.

In the absence of body force,the force in each section of the bar is constant along the bar and is denoted by $F(t)$.The compatibility condition(integral of the strains along the bar equals end displacement)gives the force displacement relationship.

$$\int_0^l \frac{F(t)}{E(1-d(l-x))}dx + \int_l^L \frac{F(t)}{E}dx = U(t) \tag{4.1}$$

Equation.4.1can be recast as:

$$\frac{F(t)}{U(t)} = E\left(\int_0^l \frac{1}{1-d(l-x)}dx + L - l\right)^{-1} \tag{4.2}$$

The factor in the right hand side of Equation.4.2 indicates the drop in stiffness of the bar as l goes from 0 to 1.We introduce the following dimensionless numbers and variables:

$$\tilde{\phi} = \tilde{l} - \tfrac{x}{l_c}, \qquad \tilde{l} = \tfrac{l}{l_c}, \qquad \tilde{l}_c = \tfrac{l_c}{L}, \qquad \tilde{F} = \tfrac{F}{\sqrt{2Y_cE}}, \qquad \tilde{U} = \tfrac{U\sqrt{E}}{L\sqrt{2Y_c}} \tag{4.3}$$

where Y_c is the critical value of the energy release rate for which damage starts.The evolution of the stiffness with $\tilde{l}$ can be derived as:

$$\frac{\tilde{F}(t)}{\tilde{U}(t)} = \left(\tilde{l}_c \int_0^{\tilde{l}} \frac{1}{1-d(\tilde{\phi})}d\tilde{\phi} + 1 - \tilde{l}\tilde{l}_c\right)^{-1} \tag{4.4}$$

We need to write the relationship between the loading and the extent of the damage l i.e. the evolution law for the front.The damage front will propagate when the following condition will be satisfied:

$$\int_0^l \frac{1}{2l_c}E\epsilon^2 = \overline{Y}_c = Y_c\frac{l}{l_c} \Rightarrow \frac{1}{l}\int_0^l \frac{1}{2}E\epsilon^2 dx = Y_c \tag{4.5}$$

Equation.4.5 signifies that the average of the local energy release over damage zone should be equal to the critical value for the front to propagate.Hence the model is called Nonlocal Model.Using the dimensionless variables as described in Equation.4.3,Equation.4.5 can be recast as:

$$\tilde{F} = \left(\frac{1}{\tilde{l}}\int_0^{\tilde{l}} \left(1-d(\tilde{\phi})\right)^{-2} d\tilde{\phi}\right)^{-\frac{1}{2}} \tag{4.6}$$

As shown in Fig.4.1(b),we consider three types of damage laws:Linear,Sine and Cosine and determine the expressions for Dimensionless Force and Displacement for each types of damage laws as shown in Table.4.1[1].

[1]Detail derivation is given in Appendix.E

Damage Law $\left(d\left(\tilde{\phi}\right)\right)$	Dimensionless Force $\left(\tilde{F}\right)$	Dimensionless Displacement $\left(\tilde{U}\right)$
$\tilde{\phi}$	$\sqrt{1-\tilde{l}}$	$\tilde{F}\left(1-\tilde{l}\tilde{l}_c-\tilde{l}_c\log\left(1-\tilde{l}\right)\right)$
$\frac{1}{2}\left(\sin\left(\pi\tilde{\phi}-\frac{\pi}{2}\right)+1\right)$	$\left(\frac{2}{\pi\tilde{l}}\left[\tan\left(\frac{\pi}{2}\tilde{l}\right)+\frac{1}{3}\tan^3\left(\frac{\pi}{2}\tilde{l}\right)\right]\right)^{-\frac{1}{2}}$	$\tilde{F}\left(1-\tilde{l}\tilde{l}_c+\frac{2\tilde{l}_c}{\pi}\tan\left(\frac{\pi\tilde{l}}{2}\right)\right)$
$1-\cos\left(\frac{\pi}{2}\tilde{\phi}\right)$	$\left(\frac{2}{\pi\tilde{l}}\tan\left(\frac{\pi\tilde{l}}{2}\right)\right)^{-\frac{1}{2}}$	$\tilde{F}\left(1-\tilde{l}\tilde{l}_c+\frac{2\tilde{l}_c}{\pi}\left[\ln\tan\left(\frac{\pi}{4}\left(1+\tilde{l}\right)\right)\right]\right)$

TABLE 4.1: Three different types of damage laws considered:Linear,Sine and Cosine.Corresponding Dimensionless Force and Displacement are computed.

4.2.2 Numerical Result and Discussion

Ref.Fig.4.1(c),we consider linear damage law and different values of $\tilde{l}_c : 0, 0.20, 0.50, 0.80, 1$.We get the snap-back response.The dissipation in the system corresponds to the area between the elastic loading and the return curves which may be computed as $Y_c l_c/2$.The higher the value of l_c,more ductile is the response.This is obvious because higher l_c means transition zone is more.

Ref.Fig.4.1(d) and (e),we have determined Force-Displacement relationship for three different types of damage laws: Linear,Sine and Cosine and for two values of l_c i.e 1 and 0.30.For lower values of l_c,the difference in response is negligible.If l_c is more,the transition zone is more and hence type of variation of damage becomes significant.

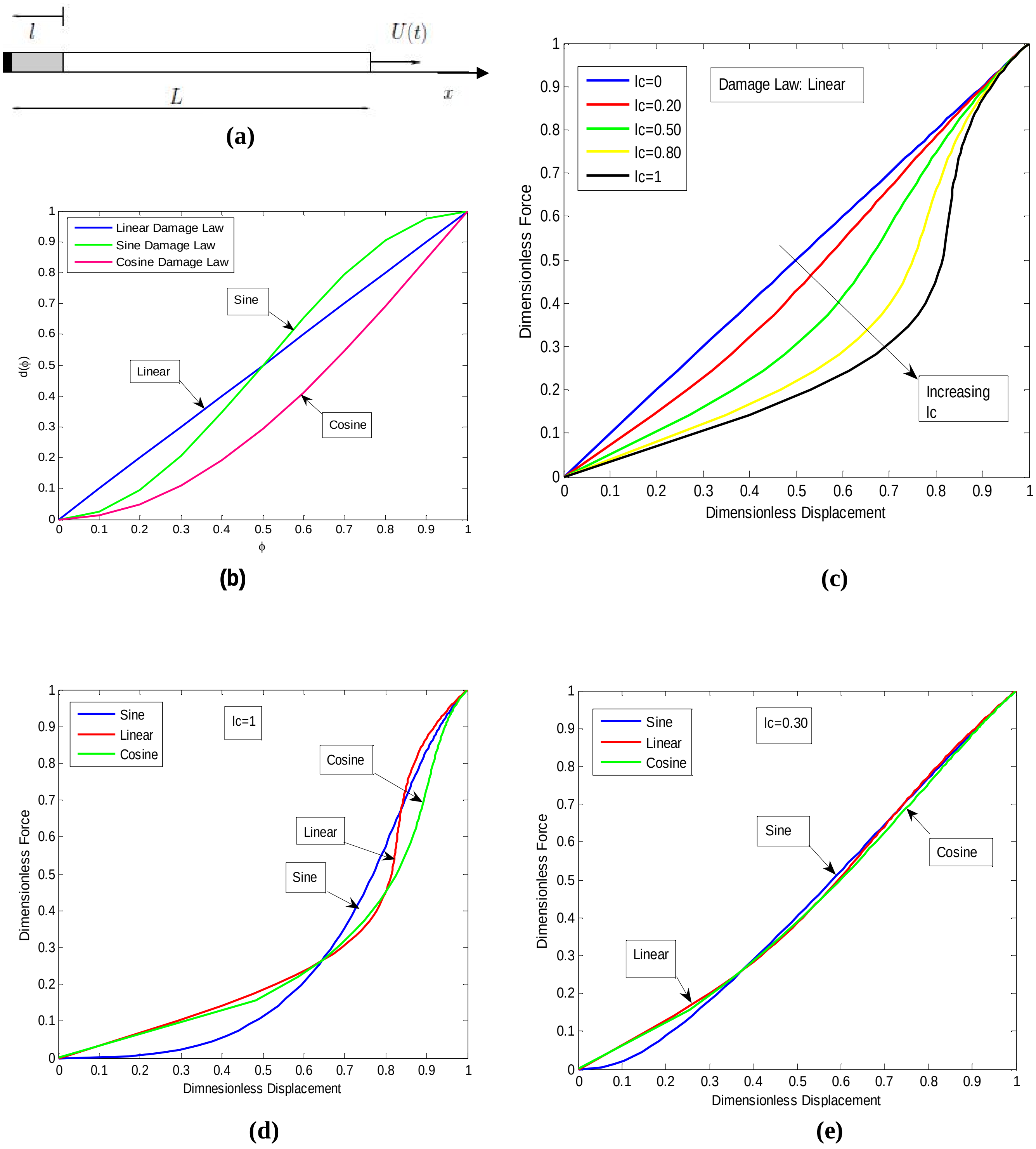


FIGURE 4.1: (a)A bar of length L subjected to some loading.Length of the damaged zone is l. (b)Three types of damage law: Linear,Sine and Cosine. (c)Load-Displacement curve for different values of $\tilde{l_c}$(Damage law is linear). (d)Load-Displacement curve for diferent damage laws for $\tilde{l_c} = 1$. (e)Load-Displacement curve for diferent damage laws for $\tilde{l_c} = 0.30$.

4.3 Force-Displacement Relationship for Two-Dimensional Case

As shown in Fig.4.2(b)or(c),we consider the circular plate with transition damage zone subjected to axysymmetric tensile loading causing propagation of damage length (l).For a given damage length,we will derive the force required to propagate the front based on the Growth Law as in Equation.1.13 and the stiffness of the system for a given damage length based on compatibility condition.Level Set ϕ can be related with the radius as : at $r = 0, \phi = l$ and $r = l, \phi = 0$.

4.3.1 Force Required to Propagate Damage Front for a Given Damage Length

Free Energy φ of the system can be obtained from Equation.1.3 as:

$$\varphi(\epsilon, d) = \frac{E\,(1 - d(r))}{2\,(1-\nu^2)}\left[\varepsilon_r^2 + 2\nu\varepsilon_r\varepsilon_\theta + \varepsilon_\theta^2\right] \tag{4.7}$$

Local Energy Release Rate can be obtained from Equation.1.2 as:

$$Y = -\frac{\partial\varphi}{\partial d} = \frac{E}{2\,(1-\nu^2)}\left[\varepsilon_r^2 + 2\nu\varepsilon_r\varepsilon_\theta + \varepsilon_\theta^2\right] \tag{4.8}$$

We will now compute the configurational force $g(s)$ as given by Equation.1.13 for this two-dimensional system.We consider the linear variation of damage i.e.$d^{'}(\phi) = l_c^{-1}$. The radius of curvature of the damage front is $\rho(s) = l$.
Changing variable ϕ in terms of r,and using Equation.1.13,we get:

$$g(s) = \int_0^l \frac{E}{2l_c\,(1-\nu^2)}\left[\varepsilon_r^2 + 2\nu\varepsilon_r\varepsilon_\theta + \varepsilon_\theta^2\right]\frac{r}{l}dr \tag{4.9}$$

We will determine the expressions of stresses in terms of external force.Finally we will determine the expression of Force in terms of the damage length.
In Chapter.2,we have determined the expression of Radial and Hoop Stress.We have studied the asympotic behavior in Chapter.3.Using these concept,we can write the expressions of strains for the present system.
For the present Chapter,we will consider the Heun Equation(Equation.2.37) and the solution by Heun function given by Equation.2.38 as:

$$\begin{aligned} u = {} & C_1 \mathrm{HeunB}\left(\sqrt{5-4k}, \sqrt{2}, 1+2k, -2k\sqrt{2}, \frac{1}{2}\frac{\sqrt{2}r_1}{r}\right) r^{-\frac{1}{2}-\frac{1}{2}\sqrt{5-4k}} \\ & + C_2 \mathrm{HeunB}\left(-\sqrt{5-4k}, \sqrt{2}, 1+2k, -2k\sqrt{2}, \frac{1}{2}\frac{\sqrt{2}r_1}{r}\right) r^{-\frac{1}{2}+\frac{1}{2}\sqrt{5-4k}} \end{aligned}$$

Note that as the damage length l propagates starting from a small damage zone, we are in different situation.When $l < l_c$, we are in the case of no void.Hence we need to use solution of this case.For the $l = l_c$,we are in a condition when $d = 1$ only at the center.Afterwards,we are in a condition when $l > l_c$ i.e. the case of void.We have to use solution for this case.
We note that when we are considering the case l in the neighbourhood of l_c i.e.when r_1 starts coming into existance and when $r_1 \to 0$,we have for any given k,
$\text{HeunB}\left(\sqrt{5-4k}, \sqrt{2}, 1+2k, -2k\sqrt{2}, 0\right) = 1$ and $\text{HeunB}\left(-\sqrt{5-4k}, \sqrt{2}, 1+2k, -2k\sqrt{2}, 0\right) = 1$.We will give the detail expressions for this case.For the other two cases i.e $l < l_c$ and $l > l_c$ we can derive the required expression in same way considering the solution for those cases.
The square of the radial strain ε_r^2 can be written as:

$$\varepsilon_r^2 = F^2 \left(\frac{C_F}{\xi - C_{R_{23}l}}\right)^2 e^{-\frac{r}{l}} \left[c_{C_1}^2 g_{r1} + c_{C_1} c_{C_2} g_{r12} + c_{C_2}^2 g_{r2}\right] \tag{4.10}$$

Here c_{C_1},c_{C_2} are given by Equation.2.70.C_F , ξ and $C_{R_{23}l}$,g_{r1} ,g_{r12}, g_{r2} are given by:

$$C_F = \frac{2lL^2}{E\left(L^2 - l^2\right)} \tag{4.11}$$

$$\xi = \frac{(1+\nu)lL^2 + (1-\nu)l^3}{E\left(L^2 - l^2\right)} \tag{4.12}$$

$$C_{R_{23}l} = c_{C_1} r^{-\frac{1}{2}\sqrt{5-4k}-\frac{1}{2}} + c_{C_2} r^{\frac{1}{2}\sqrt{5-4k}-\frac{1}{2}} \tag{4.13}$$

$$g_{r1} = \frac{1}{2r^3}\left(3r^{\sqrt{5-4k}} - 2r^{-\sqrt{5-4k}}k - r^{-\sqrt{5-4k}}\sqrt{5-4k}\right) \tag{4.14}$$

$$g_{r12} = 2(-1+k)/r^3 \tag{4.15}$$

$$g_{r2} = \frac{1}{2r^3}\left(3r^{\sqrt{5-4k}} - 2r^{\sqrt{5-4k}}k - r^{\sqrt{5-4k}}\sqrt{5-4k}\right) \tag{4.16}$$

The product of the radial and hoop strain $\varepsilon_r\varepsilon_\theta$ can be written as:

$$\varepsilon_r\varepsilon_\theta = F^2 \left(\frac{C_F}{\xi - C_{R_{23}l}}\right)^2 e^{-\frac{r}{l}} \left[c_{C_1}^2 g_{r\theta 1} + c_{C_1} c_{C_2} g_{r\theta 12} + c_{C_2}^2 g_{r\theta 2}\right] \tag{4.17}$$

$g_{r\theta 1}$,$g_{r\theta 12}$, $g_{r\theta 2}$ are given by:

$$g_{r\theta 1} = -\frac{1}{2r^3}\left(r^{-\sqrt{5-4k}}\left(1+\sqrt{5-4k}\right)\right) \tag{4.18}$$

$$g_{r\theta 12} = -\frac{1}{r^3} \tag{4.19}$$

$$g_{r\theta 2} = -\frac{1}{2r^3}\left(-r^{\sqrt{5-4k}}\left(1-\sqrt{5-4k}\right)\right) \tag{4.20}$$

The square of the hoop strain ε_θ^2 can be written as:

$$\varepsilon_\theta^2 = F^2 \left(\frac{C_F}{\xi - C_{R_{23}l}}\right)^2 e^{-\frac{r}{l}} \left[c_{C_1}^2 g_{\theta 1} + c_{C_1} c_{C_2} g_{\theta 12} + c_{C_2}^2 g_{\theta 2}\right] \tag{4.21}$$

$$g_{\theta 1} = \frac{1}{r^3} r^{-\sqrt{5-4k}} \tag{4.22}$$

$$g_{\theta 12} = \frac{2}{r^3} \tag{4.23}$$

$$g_{\theta 2} = \frac{1}{r^3} r^{\sqrt{5-4k}} \tag{4.24}$$

Critical Energy Release Rate for the 2D system can be computed using Equation.1.15 as:

$$\overline{Y}_c(s) = Y_c(s) \int_0^l Y(\phi, s) d^{'}(\phi) \left(1 - \frac{\phi}{\rho(s)}\right) d\phi = \frac{lY_c}{2l_c} \tag{4.25}$$

The configurational force/unit length given by Equation.4.9 must be equal to the Critical Energy Release Rate[2] given by Equation.4.25 in order to propagate the damage front. Equating Equation.4.9 and Equation.4.25 and using the dimensionless variables as in Equation.4.3,we get the expression for dimensionless force(per unit length along perimeter of the circular plate)required to propagate the front og damage length of l.

$$\tilde{F} = \sqrt{\frac{l}{2E}} \left(\left(\frac{C_F}{\xi - C_{R_{23}l}} \right)^2 \left(\frac{E}{(1-\nu^2)} \right) \left(\frac{1}{l} \right) \int_0^l \left[e^{-\frac{r}{l}} \left(f_{\varepsilon_r} + 2\nu f_{\varepsilon_{r\theta}} + f_{\varepsilon\theta} \right) r \right] dr \right)^{-\frac{1}{2}} \tag{4.26}$$

Where,

$$\begin{aligned} f_{\varepsilon_r} &= c_{C_1}^2 g_{r1} + c_{C_1} c_{C_2} g_{r12} + c_{C_2}^2 g_{r2} \quad f_{\varepsilon_{r\theta}} = c_{C_1}^2 g_{r\theta 1} + c_{C_1} c_{C_2} g_{r\theta 12} + c_{C_2}^2 g_{r\theta 2} \\ f_{\varepsilon_\theta} &= c_{C_1}^2 g_{\theta 1} + c_{C_1} c_{C_2} g_{\theta 12} + c_{C_2}^2 g_{\theta 2} \end{aligned} \tag{4.27}$$

4.3.2 Variation of Stiffness with Damage Propagation

It is obvious that a system with less damaged zone is more stiff.In this section,we will study how the siffness of the 2D system under consideration gets changed as the damage zone propagates.

We can write the displacemnet at the outer boundary in terms of strain of the damaged and undamaged zone as:

$$U|_{r=L} = \underbrace{\int_0^l \varepsilon_r dr}_{\text{Damaged Zone}} + \underbrace{\int_l^L \varepsilon_r dr}_{\text{Undamaged Zone}} \tag{4.28}$$

Here L is the outer radius of the circle and l is the raius of the changing damaged zone. For the damaged zone,

$$\underbrace{\int_0^l \varepsilon_r dr}_{\text{Damaged Zone}} = F \left(\frac{C_F}{\xi - C_{R_{23}l}} \right) \int_0^l \left[c_{C_1} f_{r_{11}} + c_{C_2} f_{r_{21}} \right] dr \tag{4.29}$$

[2]Note that for 1D bar case(Fig.4.1),$\overline{Y}_c = \frac{lY_c}{l_c}$,i.e.twice of the 2D case.

Where,

$$f_{r_{11}} = -\frac{1}{2r}\left(1+\sqrt{5-4k}\right) r^{-\frac{1}{2}\sqrt{5-4k}-\frac{1}{2}} \tag{4.30}$$

$$f_{r_{21}} = -\frac{1}{2r}\left(1-\sqrt{5-4k}\right) r^{\frac{1}{2}\sqrt{5-4k}-\frac{1}{2}} \tag{4.31}$$

For the undamaged zone,we can write:

$$\int_l^L \varepsilon_r dr = \frac{F}{E}\frac{1}{(L+l)}\left[Ll(1+\nu)(1-C_{fR}) + (1-\nu)\left(C_{fR}l^2 - L^2\right)\right] \tag{4.32}$$

Where $C_{fR} = C_F/(\xi - C_{R_{23}l})$.Substituting Equation.4.29 and Equation.4.32 in Equation.4.28,we get the expression of displacement in terms of force required to propagate the damage of given length l.

Further using dimensionless variables as in Equation.4.3,we can determine the dimensionless stiffness of the 2D system as a function of the propagating damage length as:

$$\tilde{K}(l) = \frac{E}{L}\left[\tilde{K}_{dam} + \tilde{K}_{undam}\right]^{-1} \tag{4.33}$$

Here $\tilde{K}_{dam}$ and $\tilde{K}_{undam}$ are the contribution to overall stiffness of the 2D system due to damaged zone and the undamaged zone respectively.

$$\tilde{K}_{dam} = C_{fR}\int_0^l \left[c_{C_1} f_{r_{11}} + c_{C_2} f_{r_{21}}\right] \tag{4.34}$$

$$\tilde{K}_{undam} = \frac{1}{E}\int_l^L \left[\frac{L^2 l^2 (1+\nu)(1-C_{fR})}{r^2 (L^2 - l^2)} + (1-\nu)\frac{C_{fR}l^2 - L^2}{(L^2 - l^2)}\right] \tag{4.35}$$

4.3.3 Numerical Result and Discussion

For the numerical result, we consider the following values:$L = 100mm$,$E = 2 \times 10^5 \frac{N}{mm^2}$,$\nu = 0.20$,$\tilde{l}_c = 0.60$.

We vary $\tilde{l}$ and for each $\tilde{l}$, we compute the length of the damage(i.e. the radius of the damage zone) l using the relation $l = \tilde{l}\cdot\tilde{l}_c\cdot L$. Ref.to Fig.4.2,we observe that the stiffness of the system gets reduced with the propagation of the damage.It is obvious because an undamaged system is more stiff as compared to a damaged one.

In Fig.4.3(a),we observe the relation between the required force to propagate damage for a given damage length.It is obvious that a system with less damaged zone requires more force to propagate damage.Form Fig.4.3(b),we observe the relationship of corresponding displacemnet with the propagation of damage.System with more damage zone yields more displacement because of reduced stiffness.

Fig.4.3(c) gives the relationship between the Force and Displacement.

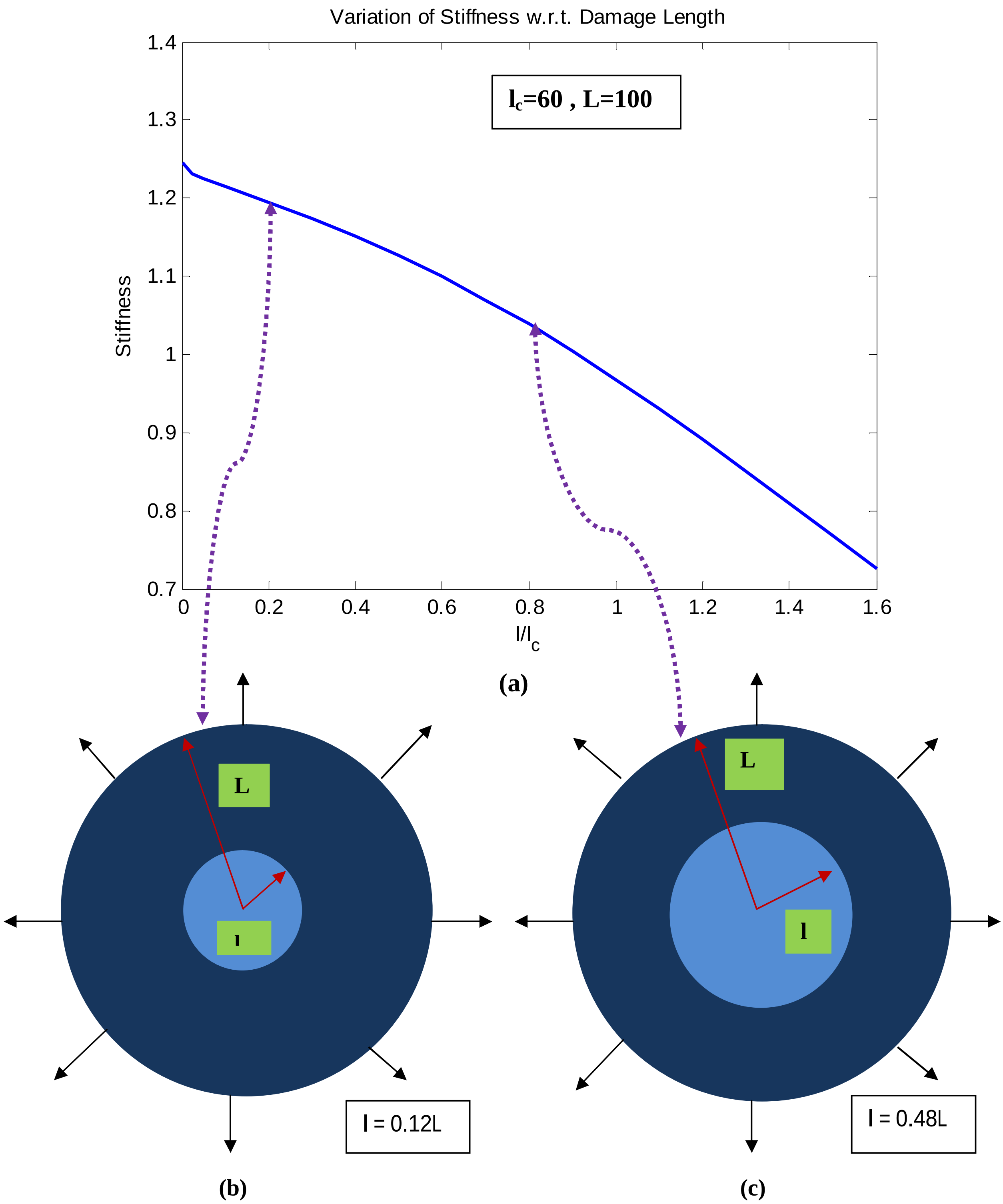


FIGURE 4.2: (a)Loss of Stiffness of the 2D circular system with concentric damaged zone with the Propagation of Damage.(b)System with less damaged zone ($l = 0.12L$) having more stiffness as compared to (c)system with more damaged zone ($l = 0.48L$).

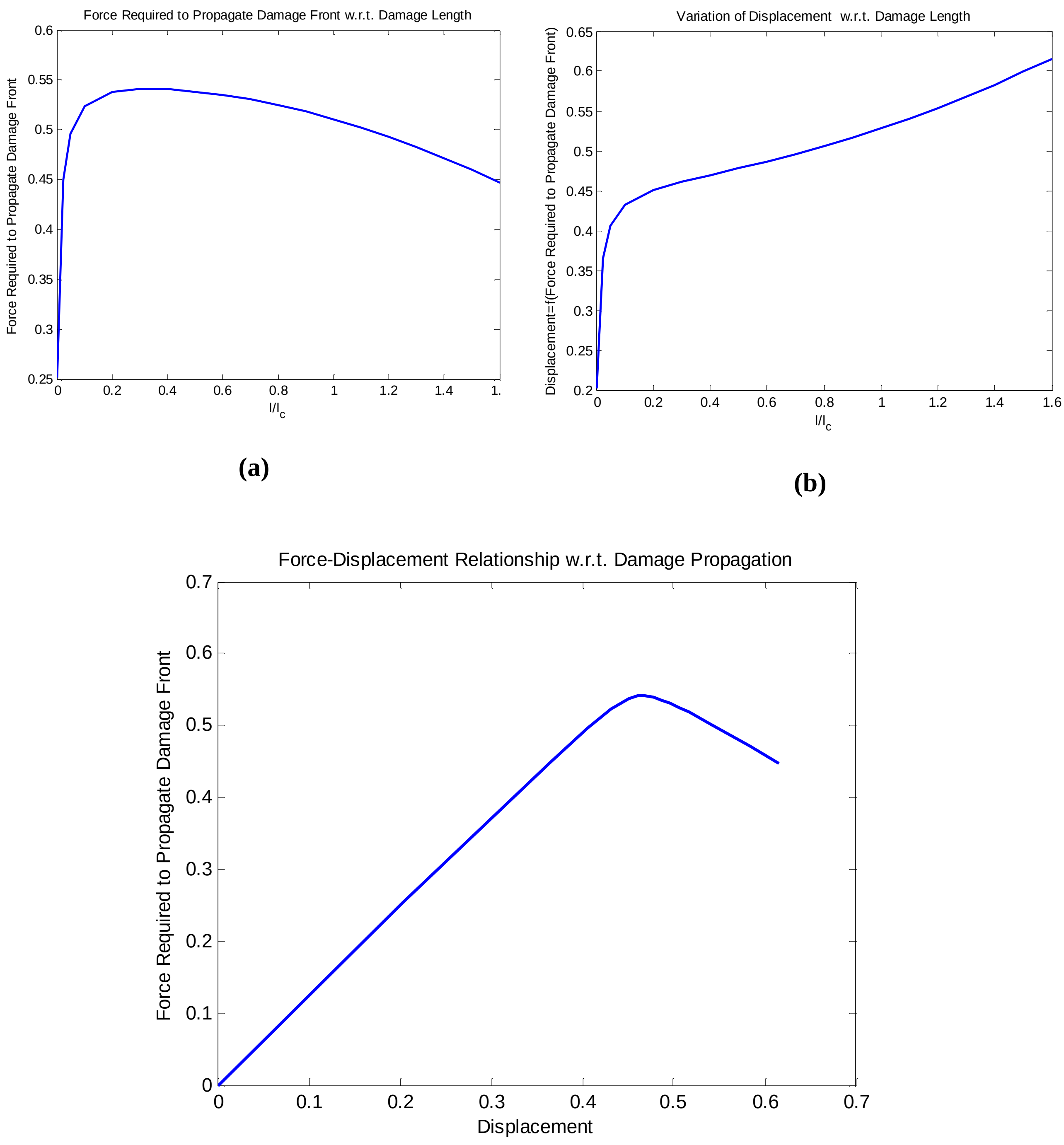


Figure 4.3: (a)Force required to propagate the damage for a given damage length of l.(b)Displacement ssociated with the corresponding force.(c)Force-Displacement relationship with damage propagation.

4.4 Algorithm to Develope Force-Displacement Relationship Using the Numerical Solution of the Analytical Equation

In Section.2.4,we have developed Finite Difference Solution of the analytical equation.Using that developement,we can compute the Force-Displacement relationship in the following ways:

- For an arbitrary value of $u|_{r=R}$ say $\tilde{U}$,compute as described in Section.2.4,the displacement,strain and stress field.
- Compute the corresponding load $(\tilde{F})$ using Equation.2.29 as:

$$\tilde{F} = f(\tilde{U}) = \frac{1}{R\tilde{U}} \int_0^R (\sigma_{rr}\varepsilon_{rr} + \sigma_{\theta\theta}\varepsilon_{\theta\theta})\, r dr \tag{4.36}$$

- Compute slope (θ) as:

$$\theta = \frac{\tilde{F}}{\tilde{U}} \tag{4.37}$$

- Compute $\lambda^2 = \frac{\tilde{F}}{F}$ using the following relation:

$$\lambda = \frac{\int_\Omega Y_c d' \, d\Omega}{\int_\Omega Y_0 d' \, d\Omega} \tag{4.38}$$

  Here Y_0 can be determined using Equation:4.8.All other notations carry the same meaning.

- Compute $F = \frac{\tilde{F}}{\lambda^2}$ and corresponding U as $U = \frac{F}{\theta}$.
- Thus we obtain one point in the Force-Displacement curve (U, F).Hence repeat the process for different values of damage length l until we get sufficient point to plot the Force-Displacement Plot.

4.5 Chapter Summary

In summery,in this chapter,we have developed anaytically Force-Displacement relationship for 1D and 2D case.In 1D case, we have considered three types of damage models:Linear,Sine and Cosine.We have studied how the different damage laws and the different values of l_c plays a role in the Force-displacemnet relationship.In 2D case,we have done the rigourous analysis to obtain the analytical expression of force required to propagate damage for a given damage length l and the stiffness of the system with damage propagation.

Chapter 5

Recommendation for Future Work

5.1 Introduction

In this closing Chapter,we will take a short tour on the future prospective of our present study.Thick Level Set Approach is new in Mechanics word.It has many many application in different aspects of mechanical problems.In this chapter,we will discuss some issues which can be addressed centered on Thick Level Set Approach e.g. Stability Analysis, Equilibrium Shape of Propagating Damage Front,Geophysical Application and for Localization Problem in Plasticity.

5.2 Topic 1 : Stability Analysis of Crack Prpagation for Thick Level Set Model

Stability-the important concept introduced by Aristotle and Aechimedes is used to study the balance of a system.Stability of the propagation of crack is significant beacuse of the technological and scientific interest.We will have a look of two types of Stability Analysis: perturbation Techniques[22] and Strain Energy Density Theory[23].

5.2.1 Perturbation Technique of Stability Analysis

5.2.1.1 Background

The inner boundary is perturbed and its effect is investigated i.e. whether the perturbation is decay with moving from boundary or not.Effort by physics and mechanics community [24] to bridge the physics of fracture to the phenomenon of fractal growth [25] has given a new taste in this topic.In some aspect,the crack propagation is similitue to the vector analogue problem to Laplacian Pattern Formation [26].It can be observed that fracture with some memory in the breaking criterion leads to fractal patterns [27].Linear Stability Analysis in [28] and [29]delivered the first information about the intrinsic instability leading to a scalar Laplacian pattern formation.

Linear stability analysis can be carried out in the following ways:First the solution with a symmetric boundary condition is considered.Then a small amplitude periodic perturbation (with wavenumber k) of the boundary is superimposed.The time dependence of the amplitude is calculated by taking into account the equation of motion of the interface and allowing deviations from the symmetric solution up to first order(since it is Linear analysis).The solution of the resulting equation will take the form $e^{\omega(k)}$,where $\omega < 0(> 0)$ means that the mode k is stable or unstable respectively.

5.2.1.2 Crack Propagation as Moving Boundary Problem

To give a general overview of this techniques,we consider the system without Thick Level Set.We can extend our study for the system with Thick Level Set.

As in Fig.5.1(a) and Fig.5.1(b),the equation of motion can be given by the Lamé Equation:

$$\nabla (\nabla \cdot u) + (1 - 2\nu)\Delta u = 0 \tag{5.1}$$

The stress $\sigma_\perp$ perpendicular to the surface for the two cases can be given by:

$$\sigma_\perp = 0 \tag{5.2}$$

$$\sigma_\perp = -p \tag{5.3}$$

The Moving Boundary Condition:

The crack surface Γ will propagate with normal growth velocity v_n which can be assumed as the function of difference between the stress $\sigma_\parallel$ parallel to the crack surface and a material dependent cohesion strength σ_c.The relation can be assumed as:

$$v_n = \frac{d\Gamma}{dt} = c\left(\sigma_\parallel - \sigma_c\right)^\eta \tag{5.4}$$

For the present study,let us assume c and η to be 1.

Problem After Perturbation:

After the perturbation at the inner boundary, the system is no longer symmetric.Problem is now asymmetric.Hence,we will have tangential componenet of displacement in addition to the radial displacement.

Denoting the derivative w.r.t.r by a prime and w.r.t.θ by a dot, the Lamé Equation 5.1 can be written as:

$$2(1-\nu)\left(u_r^{''}+\frac{u_r^{'}}{r}-\frac{u_r}{r^2}+\frac{\dot{u}_\theta^{'}}{r}-\frac{\dot{u}_\theta}{r^2}\right)+(1-2\nu)\left(\frac{\ddot{u}_r}{r^2}-\frac{\dot{u}_\theta}{r^2}-\frac{\dot{u}_\theta^{'}}{r}\right)=0 \tag{5.5}$$

$$2(1-\nu)\left(\frac{\dot{u}_r^{'}}{r}+\frac{\dot{u}_r}{r^2}+\frac{\ddot{u}_\theta}{r^2}\right)+(1-2\nu)\left(u_\theta^{''}+\frac{u_\theta^{'}}{r}-\frac{\dot{u}_r^{'}}{r}-\frac{\dot{u}_r}{r^2}-\frac{u_\theta}{r^2}\right)=0 \tag{5.6}$$

Using Hook's Law and the definition of Strain Tensor,Equation.5.4 can be written as:

$$v_n = c\left[2\nu u_r^{'}+2(1-\nu)\left(\frac{\dot{u}_\theta}{r}+\frac{u_r}{r}\right)-\sigma_c\right] \tag{5.7}$$

The set of Equations.5.5,5.6,together with the moving boundary condition (Equation.5.7) and the stress boundary conditions corresponding the problem definition represents the problem after perturbation which is analogous to the Scalar Laplacian Growth Phenomena[25].

5.2.1.3 Stability analysis of circular hole WITHOUT Thick Level Set

We will apply the concept in Section5.2.1.2 first on circular whole without Thick Level Set and then extend our study for system with Thick Level Set.Considering Fig.5.1(a),the symmetric solution can be given by using Lamé Equation.5.1 as:

$$\overline{u}_r(r) = \alpha\left[(1-2\nu)r + R_1^2 r^{-1}\right] \tag{5.8}$$

Where $\alpha = u_0/[(1-2\nu)R_0 + R_1^2 R_0^{-1}]$ and due to symmetry $\overline{u}_\theta = 0$.
At $t = 0$,we impose a perturbation of the boundary Γ as:

$$\Gamma = R_1 + \varepsilon e^{ik\theta} \tag{5.9}$$

Where ε being infinitesimally small. The solution is no longer symmetric.Hence the perturbed solution takes the form as:

$$u_r(r,\theta) = \overline{u}_r(r) + \varepsilon U_r(r)e^{ik\theta+\omega t} \tag{5.10}$$

$$u_\theta(r,\theta) = i\varepsilon U_\theta(r)e^{ik\theta+\omega t} \tag{5.11}$$

After perturbation,boundary condition given by Equation.5.2 takes the form:

$$(1-\nu)u_r^{'}+\nu\left(\frac{\dot{u}_\theta}{r}+\frac{u_\theta}{r}\right)=0 \tag{5.12}$$

$$\dot{u}_r - u_\theta + r u_\theta^{'} = 0 \tag{5.13}$$

To complete the problem after perturbation, we need to add the moving boundary condition given by Equation.5.4.

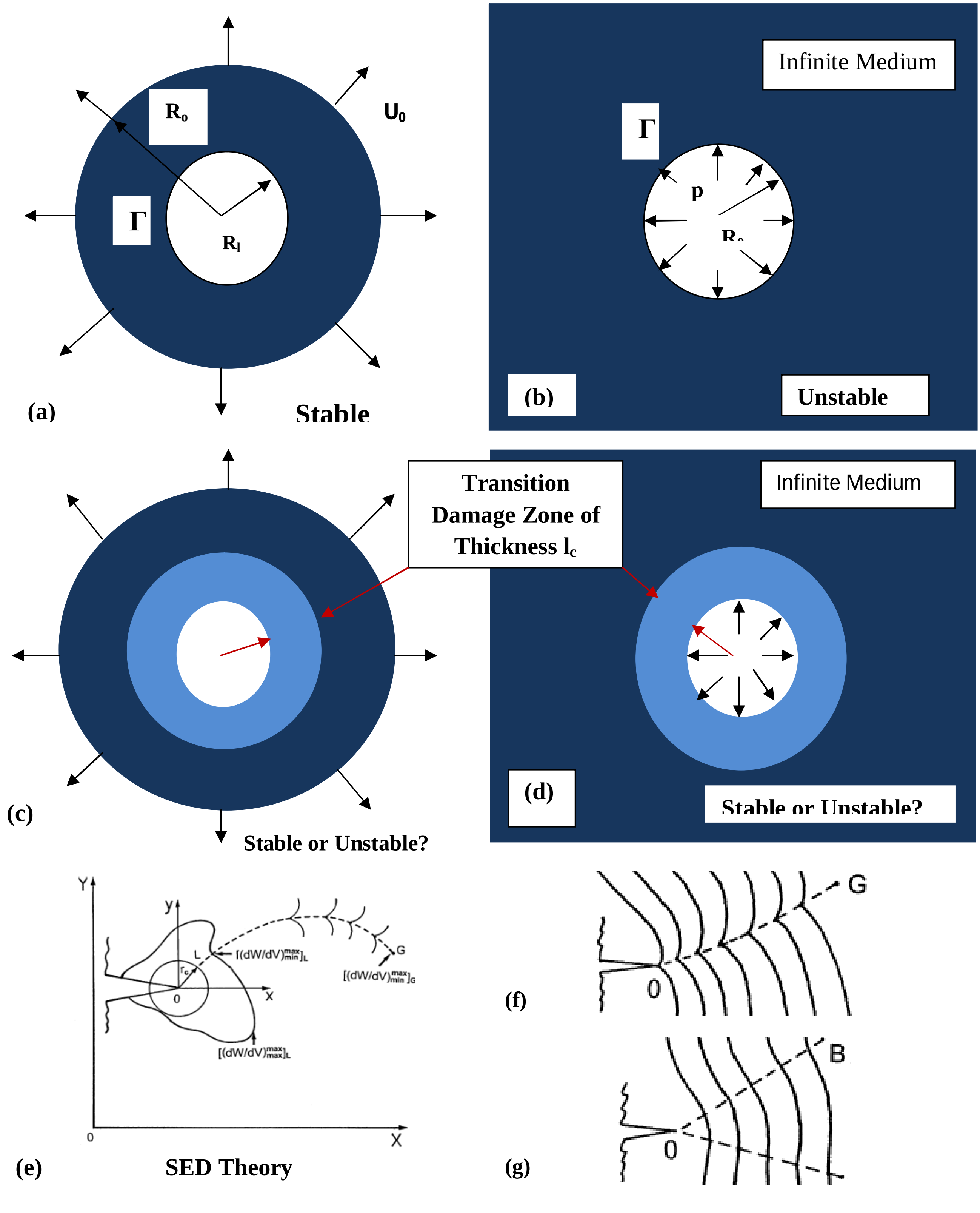


FIGURE 5.1: (a)Stretched membrane.A strain u_0 is applied at the outer circular boundary of radius R_0.The perturbation is implemented on the surface Γ of the concentric inner circle of radius R_i.(b)Pressure is applied to the surface Γ of a circle of radius R_i.The medium is taken to be infinite.Again Γ is perturbed.(c) and (d) are same system as in Fig.(a) and Fig.(b) respectively but with transition damage zone of thickness l_c. Whether the stability will be changed or not because of the transition damage zone ?.(e)SED Theory of Stability:Crack path OLG connecting the minima of strain energy density contours.(f)Stable and (g) Unstable crack path determination by energy density (dW/dV) contours.

As described in [22],the problem after perturbation leads to the solution for ω as:

$$\omega = -8C(1-2\nu)\alpha R_1 \tag{5.14}$$

As $\nu \leq 1/2$,ω can never be positive.Hence there will be no instability.Since k-dependence cancels out in the calculation,this result holds for any perturbation.
Following the same step for Fig.5.1(a),we can perform the stability analysis for the system in Fig.5.1(b).Boundary is free at infinity and a constant pressure p is applied at the inner boundary.
As described in [22],stability analysis will yield the expression of ω as:

$$\omega = \frac{4Cp}{R_1}(k-1) \tag{5.15}$$

Since $k = 1$ is only a displacement of the circle,we find that all real perturbation modes are unstable ,irrespective from the value of the Poisson ratio ν.

5.2.1.4 Stability analysis of circular hole WITH Thick Level Set

The concept mentioned in Section 5.2.1.2 can be extebded for our Thick Level Set model.Considering Fig.5.1(c) and (d),instead of Lamé eaution (5.1), the governing equilibrium equation is given by Equation.3.2.2 as:

$$\frac{\partial^2 u}{\partial r^2} + \beta_1(r)\frac{\partial u}{\partial r} + \beta_0(r)u = 0$$

Expresions of β is given in Section.2.3.2.
The governing equation of displacement for the case in Fig.5.1(c),(d) can be obtained in the same way as described in Section.2.3.2 and in Appendix.D.
The next step is to perturb the boundary Γ at $t = 0$ as in Equation.5.9 and consider the perturbed displacement similar to Equation.5.10 and Equation.5.11.To complete the perturbed problem,we need to consider the perturbed boundary conditions similar to Equation.5.12,5.13[1] and more importantly the moving boundary condition as in equation.5.4.
Then it is required to investigate how the expression of ω for cases in Fig.5.1(c) and (d) will get changed as compared to Equation.5.14 and Equation.5.15.

5.2.2 Stability Analysis by Strain Energy Density Theory

Another method for stability analysis is to compute the finite element technique to calculate the strain energy density(SED) contours.The predicted trajectory of the crack during unstable propagation is assumed to coincide with the minimum of the SED[23].

[1]Note that before perturbation,there was no tangential stress in the system.It arises after perturbation as the problem is no longer symmetric.

Strain Energy per unit volume of a material can be given by:

$$\frac{dW}{dV} = \int_0^{\varepsilon_{ij}} \sigma_{ij} d\varepsilon_{ij} = \frac{1}{2}\sigma_{ij}\varepsilon_{ij} \tag{5.16}$$

SED Theory can be summerized as:

- Fracture initiates when $[(dW/dV)^{\max}_{\min}]_L = (dW/dV)_c$
- Yielding initiates when $[(dW/dV)^{\max}_{\min}]_L = (dW/dV)_d$
- When crack starts propagate after reaching $(dW/dV)_c$,it can be simulated by finite incremental steps $r_1, r_2, \cdots .r_c$ as:

$$\left(\frac{dW}{dV}\right)_c = \frac{S_1}{r_1} = \frac{S_2}{r_2} = \cdots = \frac{S_j}{r_j} = \cdots = \frac{S_c}{r_c} \quad \text{or} \quad \frac{S_a}{r_a} \tag{5.17}$$

  Where,

  - Condition for increasing rate of crack growth leading to unstable fracture:

$$S_1 < S_2 < \cdots < S_j < \cdots < S_c \quad \text{for} \quad r_1 < r_2 < \cdots < r_j < \cdots < r_c \tag{5.18}$$

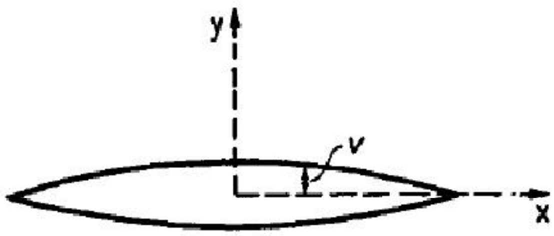
  - Condition for decreasing rate of crack growth leading to stable fracture:

$$S_1 > S_2 > \cdots > S_j > \cdots > S_a \quad \text{for} \quad r_1 > r_2 > \cdots > r_j > \cdots > r_a \tag{5.19}$$

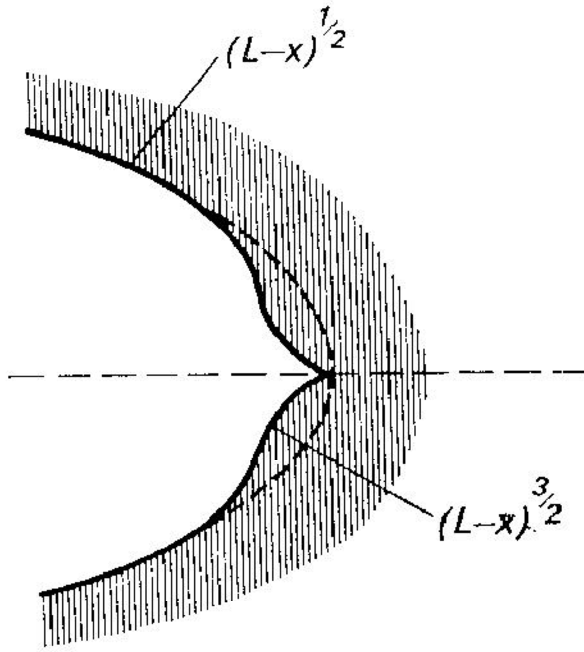
Ref Fig.5.1(f) and (g),we observe two possible patterns for the contours of the strain energy density around at the crack tip 0.In Fig.5.1(f),the *gorge* can be distinguised very clearly.This would corresponds to a stable crack path satisfying the condition given by Equation.5.19.In Fig.5.1(g),the *gorge* cannot be distinguised very clearly.For this case crack path could follow a path enclosed by the angle AOB.In Fig.5.1(e),we observe the crack trajectory given by OLG.It begins from point L and shows the maximum gradient of dW/dV.
The concepts mentioned above can be implemented for the system with Thick Level Set.

5.3 Topic 2 : Equilibrium Shape of Propagating Damage Front

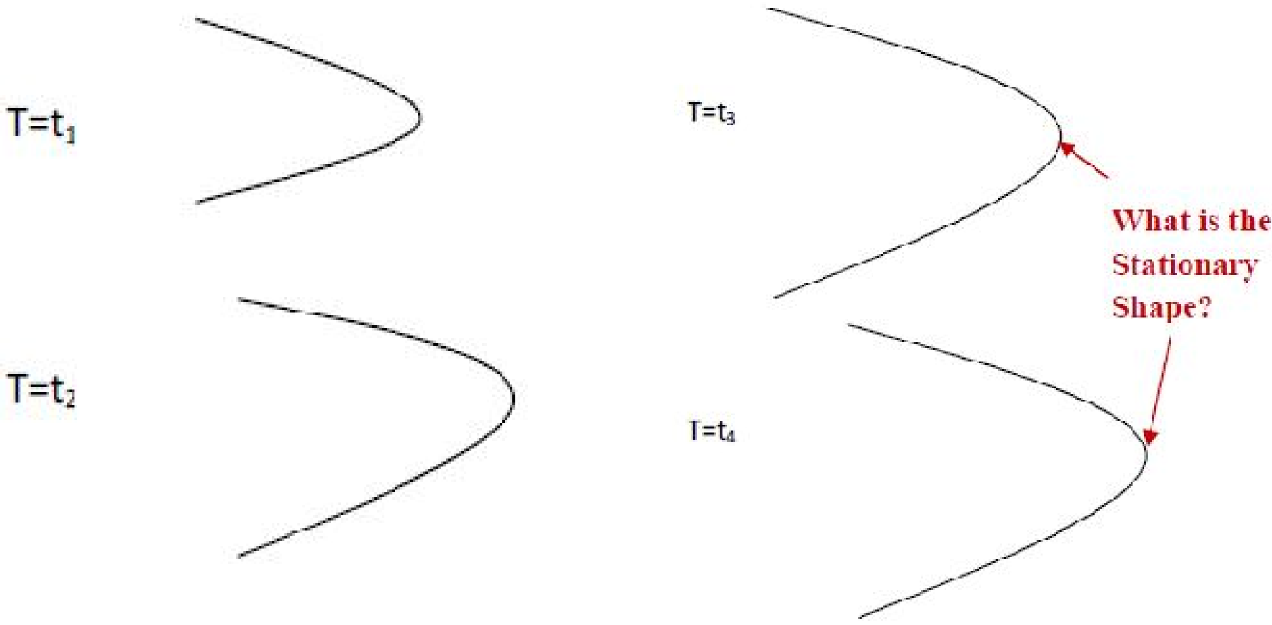
In this section, we will study the Stationary Shape of the propagating damage front. As the damage propagates, shape of the front also changes.So far no study has been done to investigate the stationary shape i.e. no change of shape of the front even with the propagation of the damage front.
We know about the equilibrium shape of crack.According to Westergaard Theory,the equilibrium shape of crack for a system subjected to Mode I loading is elliptic as shwon in Fig.5.2(a). It can be mathmatically expressed as[30]:



(a) : Equilibrium Crack Shape : Westergaard Theory



(b) Equilibrium Crack Shape: Dislocation Theory



(c) What is the stationary shape of propagating damage front?

FIGURE 5.2: (a)Elliptic crack deformation shape for Mode I loading as per Westergaard Theory.(b)Equilibrium shape of crack tip using dislocation theory.(c)Equlibrium shape of the propagating damage front.The question is not yet answered by mechanics community.

$$v = \frac{\kappa + 1}{4\mu}\sigma\sqrt{a^2 - x^2}, \quad -a \leq x \leq a. \tag{5.20}$$

Where $2a$ the length of the crack length,$\kappa = 3 - 4\nu$ for plane strain and $\kappa = (3 - \nu)/(1 + \nu)$ for gneralized plane stress.v is the component of displacement in y direction.

Nobel Lauriate Physicist Prof.L.D.Landau[31] studied the equilibrium shape of crack using the Theory of Dislocation.In Westergaard Theory[30],the shape at the crack tip follows the elliptic trajectory as shown in Fig.5.2(a).However,in dislocation theory it crack tip can be proved to have shape as shown in Fig.5.2(b).The crack length is given by $2l$.The variable width of the crack is $h(x)$ which can be given by:

$$h(x) = \int_x^l \rho(x)dx, \quad \rho(-x) = -\rho(x) \tag{5.21}$$

Using Dislocation theory, the function $\rho(x)$ can be deduced as:

$$\rho(x) = -\frac{1}{\pi^2}\sqrt{l^2 - x^2}P\int_{-l}^{l} \frac{\omega(\xi)}{\sqrt{l^2 - \xi^2}}\frac{d\xi}{\xi - x} \tag{5.22}$$

The imprtant concept in this study is to consider the very small region d where the edges of the crack join smoothly near its end and the forces of molecular attraction between the surface is taken into account.

When $l - x \approx d$,the region $l - x \approx d$ is the most important in the integral in Equation.5.22 and the Equation.5.21 can be simplified as[2]:

$$h(x) = \text{constant} \times (l - x)^{3/2} \qquad (l - x \approx d) \tag{5.23}$$

For the part farther from the end,where $d << l - x << l$,$h(x)$ can be expresed as:

$$h(x) = 2M\sqrt{l - x}/\pi^2 D \qquad (d << l - x << l) \tag{5.24}$$

The equilibrium shape can be viewd in Fig.5.2(b).

Hence we can observe that considerable study has been done on the equlibrium shape of crack in Mechanics community (e.g.Westergaard) as well in Physics community (e.g.Landau).However,no study has been done on the equlibrium shape of propagating damage front.In[1],Prof.Moës et el. showed simulation of damage propagation using Thick Level Set Approach.We can observe that the shape of the front keep changing.The open questions to the Mechanics community:Is there any stationary shape of the propagating front ? When is this stationary shape achieved ? How does the boundary conditions and external loading influence the study?

[2]Note that in order to proceed to the limit,the integral in Equation.5.22 has to be divided into two integrals with numerator $\omega(\xi) - \omega(l)$ and $\omega(l)$;the second integral makes no contribution in the limiting value as it is independent of ξ

5.4 Topic 3 : Thick Level Set for Geophysical Applications

Thick Level Set model can be extensively used for geophysical application.As an example of problems(among many!)where that could be useful,at least if it can work with inertial dynamics included, and not hinder getting to physically interesting regimes in the simulations,we can refer to work of Prof.J.Rice[32][33].
One of the major interests would be to simulate earthquake rapture propagation using Thick Level Set.Different kinds of geometric complexities e.g.step-overs,bends and branches along the faults controls the earthquake rapture propagation and in most of the cases confines the propagation extent.Study of Thick Level Set[1]can be extended to solve these kinds of geophysical problems.

5.5 Topic 4 : Thick Level Set for Localization Problem in Plasticity

The problem of localization has been the cynosure of many mechanical problems.In[34],a hypoelasto-viscoplastic endochronic model is developed to capture the strain localization phenomena.Traditional finite element model that uses the standard constitutive models suffers from excessive mesh dependency and can not reproduce the size effect commonly observed in quasi-brittle failure.In [35],a h-adaptive FEA has been developed to encounter the localization problem with reference to metal powder forming.A.Needleman studied the material rate dependence and mesh sensitivity in localization problems[36].
Thick Level Set[1]can be extended to solve these kinds of localization problems in Plasticity.

5.6 Chapter Summary

In summery, in this Chapter we have given an overview of future prospective of Thick Level Set Approach.We have investigated four different topics where Thick Level Set Approach can be extended.Apart from these topics, there are still many challenging problems in Mechanics and Physics where this level set based damage model can be extended.

Appendix A

Compatibility Equation for the Damaged Zone

Stress and Strain of the system can be related as:

$$\varepsilon_r = \frac{1}{E}(\sigma_r - \nu\sigma_\theta) \tag{A.1}$$

$$\varepsilon_\theta = \frac{1}{E}(\sigma_r - \nu\sigma_\theta) \tag{A.2}$$

$$\gamma_{r\theta} = \frac{1}{G}\tau_{r\theta} \tag{A.3}$$

We rewrite the Equation:2.15 (the Compatibility Equation for the general case in Polar coordinate):

$$\frac{\partial^2 \varepsilon_\theta}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 \varepsilon_r}{\partial \theta^2} + \frac{2}{r}\frac{\partial \varepsilon_\theta}{\partial r} - \frac{1}{r}\frac{\partial \varepsilon_r}{\partial r} - \frac{1}{r}\frac{\partial^2 \gamma_{r\theta}}{\partial r \partial \theta} - \frac{1}{r}\frac{\partial \gamma_{r\theta}}{\partial \theta} = 0$$

We write derivative of strain in terms of stress.

Using Equation:A.1,we can derive that:

$$\begin{aligned}\frac{\partial^2 \varepsilon_r}{\partial \theta^2} &= \left(\frac{1}{E}\right)\frac{\partial^2 \sigma_r}{\partial \theta^2} + \left(-\frac{2}{E^2}\frac{\partial E}{\partial \theta}\right)\frac{\partial \sigma_r}{\partial \theta} + \left(\frac{2\nu}{E^2}\frac{\partial E}{\partial \theta}\right)\frac{\partial \sigma_\theta}{\partial \theta} + \left(-\frac{\nu}{E}\right)\frac{\partial^2 \sigma_\theta}{\partial \theta^2} \\ &+ \left(\frac{\nu}{E^2}\frac{\partial^2 E}{\partial \theta^2} - \frac{2\nu}{E^3}\left(\frac{\partial E}{\partial \theta}\right)^2\right)\sigma_\theta + \left(-\frac{1}{E^2}\frac{\partial^2 E}{\partial \theta^2} + \frac{2}{E^3}\left(\frac{\partial E}{\partial \theta}\right)^2\right)\sigma_r\end{aligned} \tag{A.4}$$

$$\frac{\partial \varepsilon_r}{\partial r} = \left(-\frac{1}{E^2}\frac{\partial E}{\partial r}\right)\sigma_r + \left(\frac{1}{E}\right)\frac{\partial \sigma_r}{\partial r} + \left(\frac{\nu}{E^2}\frac{\partial E}{\partial r}\right)\sigma_\theta - \left(\frac{\nu}{E}\right)\frac{\partial \sigma_\theta}{\partial r} \tag{A.5}$$

Using Equation:A.2,we can derive that:

$$\begin{aligned}\frac{\partial^2 \varepsilon_\theta}{\partial r^2} &= \left(\frac{1}{E}\right)\frac{\partial^2 \sigma_\theta}{\partial r^2} + \left(-\frac{2}{E^2}\frac{\partial E}{\partial r}\right)\frac{\partial \sigma_\theta}{\partial r} + \left(\frac{2\nu}{E^2}\frac{\partial E}{\partial r}\right)\frac{\partial \sigma_r}{\partial r} + \left(-\frac{\nu}{E}\right)\frac{\partial^2 \sigma_r}{\partial r^2} \\ &+ \left(-\frac{1}{E^2}\frac{\partial^2 E}{\partial r^2} + \frac{2}{E^3}\left(\frac{\partial E}{\partial r}\right)^2\right)\sigma_\theta + \left(\frac{\nu}{E^2}\frac{\partial^2 E}{\partial r^2} - \frac{2\nu}{E^3}\left(\frac{\partial E}{\partial r}\right)^2\right)\sigma_r\end{aligned} \tag{A.6}$$

$$\frac{\partial \varepsilon_\theta}{\partial r} = \left(-\frac{1}{E^2}\frac{\partial E}{\partial r}\right)\sigma_\theta + \left(\frac{1}{E}\right)\frac{\partial \sigma_\theta}{\partial r} + \left(\frac{\nu}{E^2}\frac{\partial E}{\partial r}\right)\sigma_r - \left(\frac{\nu}{E}\right)\frac{\partial \sigma_r}{\partial r} \tag{A.7}$$

Using Equation:A.3,we can derive that:

$$\frac{\partial^2 \gamma_{r\theta}}{\partial r \partial \theta} = \left(\frac{-2(1+\nu)}{E^2}\frac{\partial^2 E}{\partial r \partial \theta} + \frac{4(1+\nu)}{E^3}\frac{\partial E}{\partial r}\frac{\partial E}{\partial \theta}\right)\tau_{r\theta} + \left(-\frac{2(1+\nu)}{E^2}\frac{\partial E}{\partial \theta}\right)\frac{\partial \tau_{r\theta}}{\partial r} + \left(\frac{2(1+\nu)}{E}\right)\frac{\partial^2 \tau_{r\theta}}{\partial r \partial \theta} + \left(-\frac{2(1+\nu)}{E^2}\frac{\partial E}{\partial r}\right)\frac{\partial \tau_{r\theta}}{\partial \theta} \tag{A.8}$$

$$\frac{\partial \gamma_{r\theta}}{\partial \theta} = \left(\frac{2(1+\nu)}{E}\right)\frac{\partial \tau_{r\theta}}{\partial \theta} + \left(-\frac{2(1+\nu)}{E^2}\frac{\partial E}{\partial \theta}\right)\tau_{r\theta} \tag{A.9}$$

Substituting Equation:A.4 - A.9 in the Equation:2.15,we get:

$$f_1\frac{\partial^2 \sigma_\theta}{\partial r^2} + f_2\frac{\partial \sigma_\theta}{\partial r} + f_3\frac{\partial \sigma_r}{\partial r} + f_4\frac{\partial^2 \sigma_r}{\partial r^2} + f_5\sigma_\theta + f_6\sigma_r + f_7\frac{\partial^2 \sigma_r}{\partial \theta} + f_8\frac{\partial \sigma_r}{\partial \theta} + f_9\frac{\partial^2 \sigma_\theta}{\partial \theta^2} + f_{10}\frac{\partial \sigma_\theta}{\partial \theta} + f_{11}\frac{\partial^2 \tau_{r\theta}}{\partial r \partial \theta} + f_{12}\frac{\partial \tau_{r\theta}}{\partial \theta} + f_{13}\frac{\partial \tau_{r\theta}}{\partial r} + f_{14}\tau_{r\theta} = 0 \tag{A.10}$$

where the expressions for $f_i, i = 1 \cdots 14$ are given in Equation:2.16

Stress and the Airy Stress Function can be related as[18]:

$$\sigma_r = \frac{1}{r}\frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial^2 \theta} \tag{A.11}$$

$$\sigma_\theta = \frac{\partial^2 \Phi}{\partial^2 r} \tag{A.12}$$

$$\tau_{r\theta} = \frac{1}{r^2}\frac{\partial \Phi}{\partial \theta} - \frac{1}{r}\frac{\partial^2 \Phi}{\partial r \partial \theta} \tag{A.13}$$

Using the above equations,we can derive the following equations:

$$\frac{\partial^2 \sigma_\theta}{\partial r^2} = \frac{\partial^4 \Phi}{\partial r^4} \tag{A.14}$$

$$\frac{\partial \sigma_\theta}{\partial r} = \frac{\partial^3 \Phi}{\partial r^3} \tag{A.15}$$

$$\frac{\partial \sigma_r}{\partial r} = \frac{1}{r}\frac{\partial^2 \Phi}{\partial r^2} - \frac{1}{r^2}\frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^3 \Phi}{\partial r \partial \theta^2} - \frac{2}{r^3}\frac{\partial^2 \Phi}{\partial \theta^2} \tag{A.16}$$

$$\frac{\partial^2 \sigma_r}{\partial r^2} = \frac{1}{r}\frac{\partial^3 \Phi}{\partial r^3} - \frac{2}{r^2}\frac{\partial^2 \Phi}{\partial r^2} + \frac{2}{r^3}\frac{\partial \Phi}{\partial r} - \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial r^2} - \frac{2}{r^3}\frac{\partial^3 \Phi}{\partial r \partial \theta^2} + \frac{1}{r^2}\frac{\partial^4 \Phi}{\partial r^2 \partial \theta^2} - \frac{2}{r^3}\frac{\partial^3 \Phi}{\partial r \partial \theta^2} \tag{A.17}$$

$$\frac{\partial^2 \sigma_r}{\partial \theta^2} = \frac{1}{r}\frac{\partial^3 \Phi}{\partial r \partial \theta^2} + \frac{1}{r^2}\frac{\partial^4 \Phi}{\partial \theta^4} \tag{A.18}$$

$$\frac{\partial \sigma_r}{\partial \theta} = \frac{1}{r}\frac{\partial^2 \Phi}{\partial r \partial \theta} + \frac{1}{r^2}\frac{\partial^3 \Phi}{\partial \theta^3} \tag{A.19}$$

$$\frac{\partial^2 \sigma_\theta}{\partial \theta^2} = \frac{\partial^4 \Phi}{\partial r^2 \partial \theta^2} \tag{A.20}$$

$$\frac{\partial \sigma_\theta}{\partial \theta} = \frac{\partial^2 \Phi}{\partial r^2 \partial \theta} \tag{A.21}$$

$$\frac{\partial^2 \tau_{r\theta}}{\partial r \partial \theta} = -\frac{1}{r^2}\frac{\partial^3 \Phi}{\partial r \partial \theta^2} - \frac{2}{r^3}\frac{\partial^2 \Phi}{\partial \theta^2} - \frac{1}{r}\frac{\partial^4 \Phi}{\partial r^2 \partial \theta^2} + \frac{1}{r^2}\frac{\partial^3 \Phi}{\partial r \partial \theta^2} \tag{A.22}$$

$$\frac{\partial \tau_{r\theta}}{\partial \theta} = \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2} - \frac{1}{r}\frac{\partial^3 \Phi}{\partial r \partial \theta^2} \tag{A.23}$$

$$\frac{\partial \tau_{r\theta}}{\partial r} = \frac{2}{r^2}\frac{\partial^2 \Phi}{\partial r \partial \theta} - \frac{2}{r^3}\frac{\partial \Phi}{\partial \theta} - \frac{1}{r}\frac{\partial^3 \Phi}{\partial r^2 \partial \theta} \tag{A.24}$$

Substituting Equation:A.14 to A.24 and rearranging, we obtain the Compatibility Equation for the damaged zone(Equation:2.16).

Appendix B

Equilibrium Equation for the Damaged Zone

Neglecting the body force,the equilibrium equation for the axysymmetric case can be written as:

$$\frac{\partial \sigma_r}{\partial r} + \frac{\sigma_r - \sigma_\theta}{r} = 0 \tag{B.1}$$

Stress,Strain and Displacement of the system can be related as:

$$\sigma_\theta = \frac{E}{(1-\nu^2)}\left(\varepsilon_\theta + \nu\varepsilon_r\right) \tag{B.2}$$

$$\sigma_r = \frac{E\nu}{(1-\nu^2)}\left(\varepsilon_\theta + \nu\varepsilon_r\right) + E\varepsilon_r \tag{B.3}$$

$$\varepsilon_r = \frac{\partial u}{\partial r} \tag{B.4}$$

$$\varepsilon_r = \frac{u}{r} \tag{B.5}$$

Substituting Equation:B.2 and Equation:B.3 in Equation:B.1,we obtain:

$$\frac{\partial}{\partial r}\left(\frac{E\nu}{(1-\nu^2)}\left(\varepsilon_\theta + \nu\varepsilon_r\right) + E\varepsilon_r\right) + \frac{1}{r}\left(\frac{E\nu}{(1-\nu^2)}\left(\varepsilon_\theta + \nu\varepsilon_r\right) + E\varepsilon_r - \frac{E}{(1-\nu^2)}\left(\varepsilon_\theta + \nu\varepsilon_r\right)\right) = 0 \tag{B.6}$$

First part of Equation:B.6 can be written as:

$$\frac{\partial}{\partial r}\left(\frac{E\nu}{(1-\nu^2)}\left(\varepsilon_\theta + \nu\varepsilon_r\right) + E\varepsilon_r\right) = \frac{1}{1-\nu^2}\left[\frac{\partial E}{\partial r}\left(\varepsilon_r + \nu\varepsilon_\theta\right) + E\left(\frac{\partial \varepsilon_r}{\partial r} + \nu\frac{\partial \varepsilon_\theta}{\partial r}\right)\right] \tag{B.7}$$

Second part of Equation:B.6 can be simplified as:

$$\frac{1}{r}\left(\frac{E\nu}{(1-\nu^2)}\left(\varepsilon_\theta + \nu\varepsilon_r\right) + E\varepsilon_r - \frac{E}{(1-\nu^2)}\left(\varepsilon_\theta + \nu\varepsilon_r\right)\right) = -\frac{E}{(1+\nu)r}(\varepsilon_r - \varepsilon_\theta) \tag{B.8}$$

Using Equation:B.7 and Equation:B.8,Equation:B.6 can be written as:

$$\left(\frac{E\nu}{(1-\nu^2)}\right)\frac{\partial \varepsilon_\theta}{\partial r}+\left(\frac{E}{(1-\nu^2)}\right)\frac{\partial \varepsilon_r}{\partial r}+\left(-\frac{E}{(1+\nu)r}+\frac{\nu}{1-\nu^2}\frac{\partial E}{\partial r}\right)\varepsilon_\theta+\left(\frac{E}{(1+\nu)r}+\frac{1}{1-\nu^2}\frac{\partial E}{\partial r}\right)\varepsilon_r=0 \tag{B.9}$$

Using Equation:B.4 and B.5,we get the following equations:

$$\frac{\partial \varepsilon_\theta}{\partial r}=-\frac{u}{r^2}+\frac{1}{r}\frac{\partial u}{\partial r} \tag{B.10}$$

$$\frac{\partial \varepsilon_r}{\partial r}=\frac{\partial^2 u}{\partial r^2} \tag{B.11}$$

Substituting Equations:B.4,B.5,B.10,B.11 in Equation:B.9,and rearranging,we get the Equilibrium Equation for the damaged zone(Equation:3.2.2).

Appendix C

A Quick Look at Heun's Differential Equation

Heun's Differential Equation(HDE) is a natural generalization of the Hypergeometric Equation.It is the most general linear Fuchsian Equation of second order with four regular singularities.Many problems in Mathmatical Physics can be solved using HDE as it includes the Gauss hypergeometric, confluent hypergeometric, Mathieu, Ince, Lame, Bessel, Legendre, Laguerre equations, etc.

The canonical form of Heun's general equation,whose Klein-Bocher-Ince formula[37] is $[0, 4, 0]$ is

$$u_{zz} + \left(\frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\epsilon}{z-a}\right) u_z + \frac{\alpha\beta z - q}{z(z-1)(z-a)} u = 0 \tag{C.1}$$

Note that Fuchsian Condition $\varepsilon = \alpha+\beta-\gamma-\delta+1$ is needed in order to ensure the singularity of the point at ∞.HDE has four regular singular points $z = 0, 1, a, \infty$.Every second-order linear ODE in the complex plane (or on the Riemann sphere, to be more accurate) with four regular singular points can be transformed into this equation.The solution that possesses a series expansion in the vicinity of the singular point is called Heun's function and is written H.

The Heun functions, HeunG, HeunC, HeunB, HeunD and HeunT, are defined as the solutions to the corresponding General, Confluent, Biconfluent, Doubleconfluent and Triconfluent Heun equations.

Some Important facts about the Heun Function are:

- They are more general than the rest of the functions of the mathematical language in that they contain most of them as particular cases. Consequently, the Heun equations cannot have their solution expressed (but as infinite sum power series) without using the corresponding Heun functions.

- The Heun functions have a rich structure and so satisfy a rather large number of identities.
- Because they have such a rich structure and include as particular so many functions, including the Mathieu, Lame, Spheroidal Wave and hypergeometric $2F1$,$1F1$ and $0F1$ functions, the interrelations between them and the Heun ones are a source of many nontrivial identities between the former.
- Due to the enlarged structure of singularities (if compared for instance with hypergeometric functions) the Heun functions are increasingly appearing in the modeling of different types of problems in applied mathematics.

For the present study,we have used $HeunB$ Function.The HeunB function is the solution of the Heun Biconfluent equation.The $HeunB(\alpha, \beta, \gamma, \delta, z)$ function is a local (Frobenius) solution to Heun's Biconfluent equation, computed as a power series expansion around the origin, a regular singular point. Because the next singularity is located at ∞,this series converges in the whole complex plane.
The Biconfluent Heun Equation (BHE) above is obtained from the Confluent Heun Equation (CHE) through a confluence process, that is, a process where two singularities coalesce, performed by redefining parameters and taking limits. In this case one regular singularity of the CHE is coalesced with its irregular singularity at ∞.The resulting Heun Biconfluent equation, thus, has one regular singularity at the origin, one irregular one at ∞,and includes as a particular case the $1F1$ hypergeometric confluent equation.
A special case happens when in $HeunB(\alpha, \beta, \gamma, \delta, z)$,the third parameter satisfies $\gamma = 2(n + 1) + \alpha$,where n is a positive integer.In this case the $n^{th} + 1$ coefficient in the series expansion is a polynomial of degree n in δ.When δ is a root of this polynomial,the $n^{th} + 1$ and subsequent coefficients cancel and the series truncates, resulting in a polynomial form of degree n for $HeunB$.

Appendix D

Derivation of Coefficients C_1,C_2 and Reactive Force (R_{23})

D.1 Derivation of Coefficients C_1,C_2

Radial stress for the damaged zone is given by Equation:2.46.Using Boundary Conditions for the damaged zone as described in Equation:2.47,we get the following two equations:

$$C_1C_{11} + C_2C_{21} = -R_{23} \tag{D.1}$$

$$C_1C_{12} + C_2C_{22} = 0 \tag{D.2}$$

Where C_{11},C_{21},C_{12} and C_{22} can be obtained using Equations.2.50,2.54,2.58 and 2.62 respectively.

Solving Equation:(D.1) and Equation:(D.2),we can derive the expressions for C_1 and C_2(Equation:2.48 and 2.49).

D.2 Derivation of Reactive Force (R_{23})

As described in section 2.7,the displacemnet for the damaged zone at the interface $r = r_2$ can be obtained using Equation:2.38:

$$\begin{aligned} u_{\text{dam}_{r_2}} &= R_{23}c_{C_1}\text{HeunB}\left(\sqrt{5-4k}, \sqrt{2}, 1+2k, -2k\sqrt{2}, \frac{1}{2}\frac{\sqrt{2}r_1}{r}\right) r^{-\frac{1}{2}-\frac{1}{2}\sqrt{5-4k}} \\ &\quad + R_{23}c_{C_2}\text{HeunB}\left(-\sqrt{5-4k}, \sqrt{2}, 1+2k, -2k\sqrt{2}, \frac{1}{2}\frac{\sqrt{2}r_1}{r}\right) r^{-\frac{1}{2}+\frac{1}{2}\sqrt{5-4k}} \end{aligned} \tag{D.3}$$

The terms c_{C_1} and c_{C_2} are given in Equation:2.70.

For the case when $\sigma_{rr}|_{r=r_3} = -P$,displacement for the undamaged zone at the junction $r = r_2$ can be obtained using Equation:2.5 as:

$$u_{\mathrm{undam}_{r_2}} = R_{23}\left[\frac{(1+\nu)r_2 r_3^2 + (1-\nu)r_2^3}{E\left(r_3^2 - r_2^2\right)}\right] - 2\frac{P r_2 r_3^2}{E\left(r_3^2 - r_2^2\right)} \tag{D.4}$$

Solving Equation:D.3 and D.4,we get the Equation:2.66.

For the case when $u_r|_{r=r_3} = -U$,displacement for the undamaged zone at the junction $r = r_2$ can be obtained using Equation:2.10 as:

$$u_{\mathrm{undam}_{r_2}} = R_{23}\frac{r_2\left(-1+\nu^2\right)\left(-r_3^2 + r_2^2\right)}{E\left(r_3^2(1-\nu) + r_2^2(1+\nu)\right)} - 2\frac{U r_2 r_3}{\left(r_3^2(1-\nu) + r_2^2(1+\nu)\right)} \tag{D.5}$$

Solving Equation:D.3 and D.5,we get the Equation:2.71.

Appendix E

Derivation of Force-Displacement Relationship for Different Damage Laws for 1D Case

Dimensionless force and displacement as a function of damage length can be given by Equation.4.6 and Equation.4.4 respectively as:

$$\tilde{F} = \left(\frac{1}{\tilde{l}} \int_0^{\tilde{l}} \left(1 - d(\tilde{\phi})\right)^{-2} d\tilde{\phi} \right)^{-\frac{1}{2}}$$

$$\frac{\tilde{F}(t)}{\tilde{U}(t)} = \left(\tilde{l}_c \int_0^{\tilde{l}} \frac{1}{1 - d(\tilde{\phi})} d\tilde{\phi} + 1 - \tilde{l}\tilde{l}_c \right)^{-1}$$

Linear Damage Law : $d(\tilde{\phi}) = \tilde{\phi}$

$$\tilde{F} = \left(\frac{1}{\tilde{l}} \int_0^{\tilde{l}} \left(1 - \tilde{\phi}\right)^{-2} d\tilde{\phi} \right)^{-\frac{1}{2}} = \left(\frac{1}{\tilde{l}} \frac{\tilde{l}}{1 - \tilde{l}} \right)^{-\frac{1}{2}} = \sqrt{1 - \tilde{l}} \tag{E.1}$$

$$\tilde{U}(t) = \tilde{F}(t) \left(\tilde{l}_c \int_0^{\tilde{l}} \frac{1}{1 - \tilde{\phi}} d\tilde{\phi} + 1 - \tilde{l}\tilde{l}_c \right) = \tilde{F}(t) \left(-\tilde{l}_c \log(1 - \tilde{l}) + 1 - \tilde{l}\tilde{l}_c \right) \tag{E.2}$$

Sine Damage Law : $d(\tilde{\phi}) = \frac{1}{2} \left(\sin \left(\pi\tilde{\phi} - \frac{\pi}{2} \right) + 1 \right)$

$$\tilde{F} = \left(\frac{1}{\tilde{l}} \int_0^{\tilde{l}} \left(1 - \frac{1}{2} \left(\sin \left(\pi\tilde{\phi} - \frac{\pi}{2} \right) + 1 \right) \right)^{-2} d\tilde{\phi} \right)^{-\frac{1}{2}} \tag{E.3}$$

We can write:

$$\int_0^{\tilde{l}} \left(1 - \frac{1}{2}\left(\sin\left(\pi\tilde{\phi} - \frac{\pi}{2}\right) + 1\right)\right)^{-2} d\tilde{\phi} = \int_0^{\tilde{l}} \left(1 + \cos\left(\pi\tilde{\phi}\right)\right)^{-2} d\tilde{\phi} = \frac{1}{\pi}\left[\frac{1}{6}\tan\left(\frac{\pi\tilde{\phi}}{2}\right)^3 + \frac{1}{2}\tan\left(\frac{\pi\tilde{\phi}}{2}\right)\right] \tag{E.4}$$

Substituting Equation.E.4 in Equation.E.3,we get the desired Equation of force for the case of Sine damage law as:

$$\tilde{F} = \left(\frac{2}{\pi\tilde{l}}\left[\tan\left(\frac{\pi}{2}\tilde{l}\right) + \frac{1}{3}\tan^3\left(\frac{\pi}{2}\tilde{l}\right)\right]\right)^{-\frac{1}{2}} \tag{E.5}$$

From Equation.4.4, and using the relation $\int_0^{\tilde{l}} \frac{1}{1+\cos\left(\frac{1}{2}\pi\tilde{\phi}\right)} d\tilde{\phi} = \frac{1}{\pi}\tan\left(\frac{1}{2}\pi\tilde{l}\right)$, we get Equation of displacement as:

$$\tilde{U} = \tilde{F}\left(1 - \tilde{l}\tilde{l}_c + \frac{2\tilde{l}_c}{\pi}\tan\left(\frac{\pi\tilde{l}}{2}\right)\right) \tag{E.6}$$

Cosine Damage Law : $d(\tilde{\phi}) = 1 - \cos\left(\frac{1}{2}\pi\tilde{\phi}\right)$

Following Equation.4.6 and using the relation $\int_0^{\tilde{l}} \frac{1}{\cos\left(\frac{1}{2}\pi\tilde{\phi}\right)^2} d\tilde{\phi} = \frac{1}{\pi}\tan\left(\frac{1}{2}\pi\tilde{l}\right)$,we the Equation for Froce as:

$$\tilde{F} = \left(\frac{2}{\pi\tilde{l}}\tan\left(\frac{\pi}{2}\tilde{l}\right)\right)^{-\frac{1}{2}} \tag{E.7}$$

Using Equation.4.4,we get the equation of displacement as a function of propagating damage length.

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